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DERIVATIONS OF VON NEUMANN ALGEBRAS INTO THE COMPACT IDEAL SPACE OF A SEMIFINITE ALGEBRA

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1. Introduction and statement of results. Let M be a semifinite von Neumann algebra and let $\mathscr{J}(M)$ be the norm closed two-sided ideal generated by the finite projections of M. Let $N \subseteq M$ be a subalgebra of M. A derivation of N into $\mathscr{J}(M)$ is a linear application δ: $N \mapsto \mathscr{J}(M)$ satisfying $\delta(xy) = \delta(x)y + x\delta(y)$ for $x, y \in N$. For instance, if $K \in \mathscr{J}(M)$, then the derivation $\delta(x) = (\mathrm{ad}\, K)(x) = Kx - xK$ is of this type. Such derivations implemented by elements in $\mathscr{J}(M)$ are called *inner*. There are many examples of derivations of *-subalgebras $N \subseteq M$ into the ideal $\mathscr{J}(M)$ which are not inner. A typical such example is as follows: Take $M = \mathscr{B}(\mathrm{L}^2(\mathbb{T}, \mu))$, where μ is the Lebesgue measure on the torus $\mathbb{T}$, let $N = \mathrm{C}(\mathbb{T})$ act on $\mathrm{L}^2(\mathbb{T}, \mu)$ by left multiplication, and define $\delta(x) = (\mathrm{ad}\, \mathrm{P}_{\mathrm{H}^2})(x)$, where $\mathrm{P}_{\mathrm{H}^2}$ is the projection onto the Hardy subspace $\mathrm{H}^2(\mathbb{T}, \mu)$ ([1], [11]). Then it is easy to see that $\delta(x) \in \mathscr{K}(\mathscr{H}) = \mathscr{J}(\mathscr{B}(\mathscr{H}))$ for $x \in \mathrm{C}(\mathbb{T})$ and that δ is not implemented by a compact operator.

We will, however, show in this paper that if N is self-adjoint and w-closed in M, then, except for certain situations, all derivations of N into $\mathscr{J}(M)$ are inner. Moreover, for the most typical excepted case we'll construct a counterexample.

This derivation problem was initiated in the case $M = \mathscr{B}(\mathscr{H})$ and $\mathscr{J}(M) = \mathscr{K}(\mathscr{H})$ by Johnson and Parrott in a paper of the early '70s ([3]). In that paper Johnson and Parrott wanted to characterize the commutant modulo the ideal of compact operators $\mathscr{K}(\mathscr{H}) \subseteq \mathscr{B}(\mathscr{H})$ for a von Neumann algebra $N \subseteq \mathscr{B}(\mathscr{H})$. They noted that in order to identify it with the compact perturbations of the commutant of N in $\mathscr{B}(\mathscr{H})$, it suffices to show that any derivation δ: $N \mapsto \mathscr{K}(\mathscr{H})$ is inner. They proved that this is indeed the case if N has no certain type II_1 factors as direct summands. To do this they first solved the case when N is abelian, the other cases being rather easy consequences of it. The general type II_1 case was proved recently in [7] by different techniques and using more of the ergodic theory of the type II_1 factors.

In [4] this derivation problem is studied in the more general setting when $\mathscr{B}(\mathscr{H})$ is replaced by a semifinite von Neumann algebra, $\mathscr{K}(\mathscr{H})$ by the ideal $\mathscr{J}(M)$, and the center of N is assumed to contain the center of M. Under this hypothesis it is proved that if N is either an abelian or a properly infinite von Neumann algebra, then any derivation of N into $\mathscr{J}(M)$ is inner.

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To state our results in precise terms, let us first recall that any von Neumann algebra N can be decomposed into a direct sum $N = N_0 \oplus N_1$ with N_0 a finite type I von Neumann algebra and N_1 a von Neumann algebra that has no finite type I summands. We will then say that N_0, as a subalgebra in M, is locally compatible with the center of M, $\mathscr{Z}(M)$, if there exists a partition of the unity $\{p_i\}_{i \in I}$ in the center of N_0, $\mathscr{Z}(N_0)$, so that for each i we have either $\mathscr{Z}(N_0)p_i \subseteq \mathscr{Z}(M)p_i$ or $\mathscr{Z}(M)p_i \subseteq \mathscr{Z}(N_0)p_i$.

1.1. THEOREM. *Let M be a semifinite von Neumann algebra and $\mathscr{J}(M)$ its compact ideal space. Let $N \subset M$ be a weakly closed *-subalgebra of M and suppose the finite type I summand of N is locally compatible with the center of M (in the sense described above). Then any derivation of N into $\mathscr{J}(M)$ is inner. Moreover, if δ: $N \to \mathscr{J}(M)$, then there exists $K \in \mathscr{J}(M)$, $\|K\| \leqslant 2\|\delta\|$ with $\delta = \operatorname{ad} K$. In particular, the commutant modulo $\mathscr{J}(M)$ of N in M equals $N' \cap M + \mathscr{J}(M)$.*

Thus, Theorem 1.1 solves in the affirmative the derivation problem if N is of type II_1 or properly infinite. It also gives an affirmative answer to the remaining case when N is finite of type I (e.g., when N is the tensor product of a matrix algebra with an abelian algebra) under an additional assumption of local compatibility between the centers of N and M. The typical situation when this condition is not fulfilled is when N is abelian and diffuse (i.e., without atoms), $\mathscr{Z}(M)$ is also diffuse, and N and $\mathscr{Z}(M)$ are independent von Neumann algebras, namely, N and $\mathscr{Z}(M)$ generate the von Neumann algebra $N \overline{\otimes} \mathscr{Z}(M)$ with N, $\mathscr{Z}(M)$ sitting inside it as $N \otimes 1$ and $1 \otimes \mathscr{Z}(M)$.

The second theorem that we will prove in this paper deals with the most simple such case, left open by Theorem 1.1, namely, when $M = \mathrm{L}^\infty([0,1], \lambda) \overline{\otimes} \mathscr{B}(\mathrm{L}^2(\mathbb{T}, \mu))$ and $N = 1 \otimes \mathrm{L}^\infty(\mathbb{T}, \mu)$. In this case we will construct a counterexample, showing the existence of a derivation of N into $\mathscr{J}(M)$ not implemented by an element in $\mathscr{J}(M)$. This is somehow unexpected and is probably the first nonvanishing 1-cohomological result in von Neumann algebras. It practically shows that the one-parameter version of Johnson and Parrott's original result may fail to be true. In order to have an alternative, more intuitive interpretation of the next theorem, the reader should notice that we may identify $M = \mathrm{L}^\infty([0,1], \lambda) \overline{\otimes} \mathscr{B}(\mathrm{L}^2(\mathbb{T}, \mu))$ with $\mathrm{L}^\infty([0,1], \mathscr{B}(\mathrm{L}^2(\mathbb{T}, \mu)))$, $\mathscr{J}(M)$ with $\mathrm{L}^\infty([0,1], \mathscr{K}(\mathrm{L}^2(\mathbb{T}, \mu)))$, and $N = 1 \otimes \mathrm{L}^\infty(\mathbb{T}, \mu)$ with the set of constant $\mathrm{L}^\infty(\mathbb{T}, \mu))$ valued functions on the interval $[0,1]$.

1.2. THEOREM. *Let $M = L^\infty([0,1], \lambda) \overline{\otimes} \mathscr{B}(L^2(\mathbb{T}, \mu))$, $N = 1 \otimes L^\infty(\mathbb{T}, \mu) \subset M$. There exists an operator $T \in M$ which commutes modulo $\mathscr{J}(M)$ with all the elements in N but which is not a compact (i.e., $\mathscr{J}(M)$) perturbation of an element commuting with M. In particular, there exists a derivation $\delta (= \operatorname{ad} T)$ from N into $\mathscr{J}(M)$ which is not inner, i.e., not implemented by an element in $\mathscr{J}(M)$.*

The paper is organized as follows: In sections 2–7 we prove Theorem 1.1 and in section 8 we prove Theorem 1.2. We will now present some of the ideas behind the proof of Theorem 1.1.

A key idea of our proof is to work with a new norm on the algebra M, denoted $|||\ |||$, which in our problem turns out to be the right correspondent of the uniform norm on $\mathscr{B}(\mathscr{H})$. This norm has two main features: it helps one deal with the center of M when diffuse and with the continuous dimension of projections when M is of type II_∞. The definition and main properties of the norm $|||\ |||$ are discussed in section 2.

We then prove Theorem 1.1 in the case N is atomic and abelian. In the proof we define the operator implementing δ as $\Sigma_i \delta(e_i)e_i$, where e_i are the atoms of N and the series is strongly convergent, and we use an adaptation of a trick in [3] to show that $\Sigma_i \delta(e_i)e_i \in \mathscr{J}(M)$.

By the atomic abelian case and by the same argument as in 4.1 [7] (for $M = \mathscr{B}(\mathscr{H})$) we prove a continuity result, namely, that if N is finite and countably decomposable, then δ is continuous from the unit ball of N with the strong operator topology into $\mathscr{J}(M)$ with the norm $|||\ |||$. Using this result, we prove that, in most situations, if an element T is in $\mathrm{K} = \overline{\mathrm{co}}^{\mathrm{w}}\{\delta(u)u^*|u$ unitary element in $N\} \subset M$ and implements δ on N, then it is in $\mathscr{J}(M)$. From this we easily get the proof of the theorem for finite type I (under the local compatibility condition) and properly infinite algebras and also reduce the remaining type II_1 case to the situation when N is separable and M is countably decomposable. Moreover, by using the Ryll-Nardzewski fixed point theorem in the same way it is used to prove the Kadison-Sakai theorem on derivations of von Neumann algebras, we make the reduction to the case when $N' \cap M$ contains no finite projections of M.

Finally, we prove the type II_1 case under the above assumptions: To construct a candidate for the operator $K \in \mathscr{J}(M)$ implementing δ on N, we show that N has an approximately finite-dimensional type II_1 von Neumann subalgebra $R \subset N$ which contains a maximal abelian *-subalgebra A of N such that $A' \cap M$ contains no finite projections of M. The proof of this fact is inspired by [6].

We then deduce that there exists $K \in \mathscr{J}(M)$ implementing δ on A, and the rest of the proof shows that in fact this K implements δ on all N. To this end we proceed by contradiction, following the lines of the proof in [7]. The assumption $\delta_0 = \delta - \operatorname{ad} K \neq 0$ shows that $\delta_0(v) \neq 0$ for some unitary element $v \in N$. Then, with the help of A and v and using some technical devices similar to 2.1 in [7], we construct a sequence of abelian subalgebras A_n in N on which δ_0 behaves as badly as possible. More precisely, we construct the algebras A_n together with some finite projections $e_n \in M$ so that if we consider M as acting on $\mathrm{L}^2(M, \varphi)$, then the compressions of $\delta_0|A_n$ to the spaces $\overline{A_n e_n} \subset \mathrm{L}^2(M, \varphi)$ are spatially isomorphic to a sequence of derivations $\delta_n\colon \mathrm{L}^\infty(\mathbb{T}, \mu) \mapsto \mathscr{B}(\mathrm{L}^2(\mathbb{T}, \mu))$. We do this in such a way that the derivations δ_n behave more and more like $\operatorname{ad} \mathrm{P}_{\mathrm{H}^2}$ and, moreover, so that by the continuity result the limit $\operatorname{ad} \mathrm{P}_{\mathrm{H}^2}$ follows so-normic continuous. This is easily seen to be a contradiction. We mention that the construction of the finite projections e_n, which doesn't appear in [7], is essential here and carries most of the technical difficulties of passing from the case $M = \mathscr{B}(\mathscr{H})$ to the general case. In fact the reader will note that, although the proof of Theorem 1.1 is inspired in certain places by [3] and [7], our approach is rather new even when particularized to the case $M = \mathcal{B}(\mathcal{H})$.

It is our feeling that the new techniques we introduce here to deal with the case M is of type II_∞ may also be used to prove Voiculescu or Andersen stability type theorems obtained when replacing $\mathcal{B}(\mathcal{H})$ by a type II_∞ factor M and $\mathcal{K}(\mathcal{H})$ by $\mathcal{J}(M)$.

As the referee of the first version of this paper pointed out to us, the paper contained an error in one of the preliminary considerations, a fact that actually made that proof of Theorem 1.1 correct only in the case where the semifinite algebra M had atomic center. We deeply thank the referee for pointing this out to us. However, in order to make the proof of 1.1 work in the generality presented in this paper, we only had to modify the definition of the norm $|||\ |||$ and to adapt accordingly some of the statements and proofs in the preliminary section 2, a matter that only affected their form, not their spirit. In turn, the fact that in certain situations the problem has a negative answer seems to us of even more interest and clearly deserves further investigation. In particular, our Theorem 1.2 shows that one-parameter versions of classical derivation problems (or higher cohomological problems) may have negative answers.

2. Some preliminaries.

2.1. Let M be a semifinite von Neumann algebra. Assume M has countable decomposable (or countable type) center $\mathcal{Z}(M)$ and let ψ be a normal faithful state on $\mathcal{Z}(M)$, fixed from now on. We will associate to ψ a normal semifinite faithful trace φ on M in the following way:

Let M be decomposed into a direct sum as $M = \oplus_{i \in I}(M_i \otimes \mathcal{B}(\mathcal{H}_i))$, where M_i are finite von Neumann algebras and $\dim \mathcal{H}_i \neq \dim \mathcal{H}_j$ for $i \neq j$. Let $\mathcal{Z}_i = \mathcal{Z}(M_i)$ be the center of M_i. Then $\mathcal{Z}(M)$ is naturally isomorphic to $\oplus \mathcal{Z}_i$. On each M_i there is a unique normal finite faithful trace φ_i which equals φ when restricted to $\mathcal{Z}_i$ (here $\mathcal{Z}_i$ is regarded as a subalgebra of $\oplus \mathcal{Z}_i = \mathcal{Z}(M)$ in the obvious way). Thus there exists a unique normal semifinite faithful trace φ on M which equals $\varphi_i \otimes \mathrm{Tr}$ on $M_i \,\overline{\otimes}\, \mathcal{B}(\mathcal{H}_i)$, where Tr is the usual trace on $\mathcal{B}(\mathcal{H}_i)$.

We denote $M_\varphi = \{x \in M | \varphi(x^*x) < \infty\}$ and, for $x \in M$, $\|x\|_\varphi = \varphi(x^*x)^{1/2}$. Let $\mathcal{H}_\varphi$ be the Hilbert space completion of M_φ in the norm $\|\ \|_\varphi$. M will always be regarded in its standard representation, acting on $\mathcal{H}_\varphi$ by left multiplication. The usual uniform norm of an operator in M will be denoted $\|\ \|$.

Note that if $e \in M$ is a finite projection, then we do not necessarily have $e \in M_\varphi$ (actually, this implication holds true only in the case where the properly infinite part of M has finite-dimensional center). However, we clearly have

2.1.1. If $e \in M$ is an finite projection, then there is an increasing sequence of central projections $p_n \in \mathcal{Z}(M)$, so that $p_n \uparrow 1$ and $ep_n \in M_\varphi$ for all n.

2.2. We set $M^1_{\varphi,\psi} = \{x \in M_\varphi | \|x\| \leqslant 1, \varphi(x^*xp) \leqslant \psi(p) \text{ for all } p \in \mathcal{Z}(M)\}$. Although we will not use any reduction theory argument in this paper, it may be helpful for the reader to note that if M is regarded as a measurable field of (semifinite) von Neumann factors, then, roughly speaking, a projection is in $M^1_{\varphi,\psi}$ if in each point it has dimension $\leqslant 1$.

The following properties of $M^1_{\varphi,\psi}$ will be frequently used:

2.2.1. If $T \in M$, $\|T\| \leqslant 1$, then $TM^1_{\varphi,\psi} \subset M^1_{\varphi,\psi}$ and $M^1_{\varphi,\psi}T \subset M^1_{\varphi,\psi}$.
2.2.2. If e_0, e are projections in M with $e_0 \prec e$ and $e \in M^1_{\varphi,\psi}$, then $e_0 \in M^1_{\varphi,\psi}$.
2.2.3. If $x \in M^1_{\varphi,\psi}$, then $x^* \in M^1_{\varphi,\psi}$ and $|x| \in M^1_{\varphi,\psi}$.
2.2.4. If f is a nonzero projection in M, then there exists a projection $e_0 \neq 0$ in $M^1_{\varphi,\psi}$ with $e_0 \leqslant f$. If, in addition, e is properly infinite with central support p, then e_0 may be chosen so as to have central support p and so that $\varphi(e_0) = \psi(p)$.

Properties 2.2.1–2.2.3 are trivial consequences of the definitions. To prove 2.2.4 it is sufficient to consider the case $M = M_0 \overline{\otimes} \mathscr{B}(\mathscr{H})$, where M_0 is finite with center $\mathscr{Z}(M_0) = \mathscr{Z}(M) = \mathscr{Z}$, ψ is a normal faithful state on $\mathscr{Z}$, and $\varphi = \tau \otimes \mathrm{Tr}$, where τ is the unique trace on M_0 which equals ψ when restricted to $\mathscr{Z}(M_0) = \mathscr{Z}$. Let e_0' be a minimal projection of $\mathscr{B}(\mathscr{H})$ and $e_0 = 1 \otimes e_0'$. By the comparison theorem there exists a central projection $p \in \mathscr{Z}$ such that $e_0 p \prec fp$ and $e_0(1-p) \succ f(1-p)$. Thus, in particular, if f is properly infinite, then $f(1-p) = 0$ so that $0 \neq e_0 p \prec fp = f$ and in fact p equals the central support of f (because e_0 has central support one). Thus we always have a nonzero projection e under f in $M^1_{\varphi,\psi}$, and if, in addition, f is properly infinite, then e may be chosen to have the trace equal to the trace of $e_0 p$, i.e., $\varphi(e) = \varphi(e_0 p) = \psi(p)$.

2.3. *Definition.* For $T \in M$ we put $|||T||| = \sup\{\|Tx\|_\varphi | x \in M^1_{\varphi,\psi}\}$. This is clearly a norm on M. It will play an important role in the sequel. Note that $|||T||| \leqslant \|T\|$ and that the equality holds if $M = \mathscr{B}(\mathscr{H})$ but fails if M is nonatomic.

The next few properties are easy consequences of the definitions and of the properties of $M^1_{\varphi,\psi}$.

2.3.1. If $T_1, T_2, T \in M$, then $|||T_1 T T_2||| \leqslant \|T_1\|\,|||T|||\,\|T_2\|$ and $|||T||| = |||T^*||| = |||\,|T|\,|||$.

2.3.2. If $T \in M$ and $\{p_n\}_n$ are disjoint central projections in M, then $|||T\Sigma_n p_n|||^2 = \Sigma_n |||Tp_n|||^2$.

2.3.3. If $T, T_0 \in M$, $\|T_0\| \leqslant 1$, and $|||T||| = |||TT_0|||$, then $|||Tp||| = |||TT_0 p|||$ for any central projection $p \in \mathscr{Z}(M)$.

2.3.4. If $f \in M$ is a properly infinite projection with central support $p \in \mathscr{Z}(M)$, then $|||f||| = \psi(p)^{1/2}$.

2.4. We denote by $\mathscr{J}(M)$ the norm closed two-sided ideal of M generated by the finite projections of M. Thus an element $T \in M$ is in $\mathscr{J}(M)$ if and only if all the spectral projections $E_{[t,\infty)}(|T|)$ of $|T|$, corresponding to intervals $[t,\infty)$ with

$t > 0$, are finite projections. Alternatively, $\mathcal{J}(M)$ may be characterized as follows:

2.4.1. $K \in \mathcal{J}(M)$ if and only if, given any $\varepsilon > 0$, there is a $K_0 \in M_\varphi$ and a projection $p_0 \in \mathcal{Z}(M)$ such that $\psi(p_0) \geqslant 1 - \varepsilon$, $K_0 p_0 = K_0$ and $\|Kp_0 - K_0\| < \varepsilon$.

Indeed, assume $K \in \mathcal{J}(M)$ and let $e = E_{[\varepsilon, \infty)}(|K|)$. Then e is a finite projection of M, so that by 2.1.1 there exists a projection $p_0 \in \mathcal{Z}(M)$ so that $\psi(p_0) \geqslant 1 - \varepsilon$ and $\varphi(ep_0) < \infty$. Let $K_0 = Kep_0$. Then $\varphi(K_0^* K_0) \leqslant \|K\|^2 \varphi(ep_0) < \infty$ and clearly $\|Kp_0 - K_0\| < \varepsilon$.

The other implication is trivial and, in fact, will not be needed in the sequel.

2.5. Let $K \in \mathcal{J}(M)$ and $\{e_n\}_n$ be a sequence of mutually orthogonal projections in M. If $M = \mathcal{B}(\mathcal{H})$, then it follows that $\|Ke_n\| \mapsto 0$ and $\|e_n K\| \mapsto 0$. This is no longer true for general M, but still we have $|||Ke_n||| \mapsto 0$ and $|||e_n K||| \mapsto 0$. Indeed, to prove this, since K is a linear combination of four positive elements in $\mathcal{J}(M)$, we may assume K is positive and $K \leqslant 1$. Let $\varepsilon > 0$ and $p \in \mathcal{Z}(M)$, $K_0 \in \mathcal{J}(M)_+$ such that $\varphi(K_0^2) < \infty$, $\|Kp - K_0\| < \varepsilon/3$, and $\psi(p) \geqslant 1 - \varepsilon/3$ as in 2.4.1. Since e_n tend weakly to zero, we have $\|K_0 e_n\|_\varphi^2 = \|e_n K_0\|_\varphi^2 = \varphi(e_n K_0^2) \mapsto 0$. But if $x \in M^1_{\varphi,\psi}$, then we have

$$\|Ke_n x\|_\varphi \leqslant \|K_0 e_n x\|_\varphi + \|(K - K_0) e_n p x\|_\varphi + \|(K - K_0) e_n (1-p) x\|_\varphi$$

$$\leqslant \|K_0 e_n\|_\varphi + \|Kp - K_0\| + \psi(1-p) \leqslant \|K_0 e_n\|_\varphi + 2\varepsilon/3,$$

so that if n is big enough, then $\|Ke_n x\|_\varphi < \varepsilon$ independently on $x \in M^1_{\varphi,\psi}$. Thus $|||Ke_n||| \mapsto 0$ and, similarly, $|||e_n K||| \mapsto 0$.

2.6. If $T \in M$, we denote by $|||T|||_{\text{ess}} = \inf\{|||T - K||| \mid K \in \mathcal{J}(M)\}$. Note that if $T \notin \mathcal{J}(M)$, then $|||T|||_{\text{ess}} > 0$. Indeed, if $T \notin \mathcal{J}(M)$, then there exists $t > 0$ such that $E_{[t,\infty)}(|T|)$ is an infinite projection. Thus there exists a sequence of mutually orthogonal, mutually equivalent infinite projections $\{f_n\}_n$ with $0 \neq f_n \leqslant E_{[t,\infty)}(|T|)$. For each n we take $0 \neq e_n \leqslant f_n$, $e_n \in M^1_{\varphi,\psi}$, e_n mutually equivalent. Thus if $K \in \mathcal{J}(M)$, we get by 2.5 $|||T - K||| \geqslant \lim_n \sup \|(T - K)e_n\|_\varphi = \lim_n \sup \|Te_n\|_\varphi \geqslant t\|e_1\|_\varphi$, which shows that $|||T|||_{\text{ess}} \geqslant t\|e_1\|_\varphi > 0$.

In fact, in certain simple situations this norm can be computed.

2.6.1. If f is a properly infinite projection of central support p, then $|||f|||_{\text{ess}} = \psi(p)^{1/2} = |||f|||$. More generally, if $T \in M$, $T \geqslant 0$ is of the form $T = \sum_{i=1}^n c_i f_i$ for some $c_i > 0$ and properly infinite mutually orthogonal projections f_i of the same central support p, then $|||T|||_{\text{ess}} = (\max\{c_i\})\psi(p)^{1/2}$.

Indeed we have $c_k f_k \leqslant T \leqslant (\max\{c_i\})\sum_j f_j$ for all k, which shows that the first part implies the second. Now the first part follows by taking a sequence of mutually orthogonal, mutually equivalent projections $e_n \in M^1_{\varphi,\psi}$ under f so that each e_n has central support p and so that $\varphi(e_n) = \psi(p)$. Then for any $K \in \mathscr{J}(M)$ we get $|||f - K||| \geqslant \lim_n \sup \|(f - K)e_n\|_\varphi = \psi(p)^{1/2}$.

Let us also note that we have for the norm $||| \ |||_{\text{ess}}$ similar properties as the properties 2.3.2 and 2.3.3 of the norm $||| \ |||$.

2.6.2. If $T, T_0 \in M$, $\|T_0\| \leqslant 1$, and $|||T|||_{\text{ess}} = |||TT_0|||_{\text{ess}}$, then $|||Tp|||_{\text{ess}} = |||TT_0p|||_{\text{ess}}$ for any central projection $p \in \mathscr{Z}(M)$.

2.6.3. If $\{p_n\}_n$ are disjoint central projections in M, then $|||T\Sigma p_n|||^2_{\text{ess}} = \Sigma|||Tp_n|||^2_{\text{ess}}$.

2.7. The norms $||| \ |||$ and $||| \ |||_{\text{ess}}$ will play a similar role in this paper as the uniform and usual essential norms do in the proof of the case $M = \mathscr{B}(\mathscr{H})$ in [3] and [7]. For our general problem these norms have all the advantages but one: for two operators of disjoint right and left supports the norm of their sum does not equal the maximum of their norms (as do the uniform and usual essential norms for $M = \mathscr{B}(\mathscr{H})$). We will instead use the following weaker property of the norm $||| \ |||_{\text{ess}}$.

LEMMA. *If $T \in M$ and f_1, f_2 are mutually orthogonal projections in M, then there are central projections $p_1, p_2 \in \mathscr{Z}(M)$ such that $p_1 + p_2 = 1$ and $|||f_iTf_ip_i|||_{\text{ess}} = |||(f_1Tf_1 + f_2Tf_2)p_i|||_{\text{ess}}$, $i = 1, 2$.*

Proof. Let's first show that if $T_1, T_2 \in M$, then there exists a central projection $p \in \mathscr{Z}(M)$ such that $|||qT_1|||_{\text{ess}} \geqslant |||T_2q|||_{\text{ess}}$ for any projection $q \in \mathscr{Z}(M)$, $q \leqslant p$, and $|||T_1q|||_{\text{ess}} \leqslant |||T_2q|||_{\text{ess}}$ for any projection $q \in \mathscr{Z}(M)$, $q \leqslant 1 - p$. Indeed, by 2.6.3 it follows that there exists a maximal projection p in $\mathscr{Z}(M)$ so that for any $q \leqslant p$, $q \in \mathscr{P}(\mathscr{Z}(M))$, we have $|||T_1q|||_{\text{ess}} \geqslant |||T_2q|||_{\text{ess}}$. Now if for some $q_0 \leqslant 1 - p$ we have $|||T_1q_0|||_{\text{ess}} > |||T_2q_0|||_{\text{ess}}$, then there exists some $p_0 \leqslant q_0$ so that for any $q \leqslant p_0$ we have $|||T_1q|||_{\text{ess}} > |||T_2q|||_{\text{ess}}$; otherwise, by 2.6.3 and a maximality argument, we get a contradiction. Thus $|||T_1q||| \leqslant |||T_2q|||$, for any $q \leqslant 1 - p$.

Now we have that if $T_1 = f_1Tf_1$, $T_2 = f_2Tf_2$, and p is the central projection corresponding to T_1, T_2 as above, then $p_1 = p$, $p_2 = 1 - p$, will satisfy the conditions. To show this, note first that since $|f_1Tf_1 + f_2Tf_2| = |f_1Tf_1| + |f_2Tf_2|$, by 2.3.1 it follows that it is sufficient to prove this assertion in the case $T_1, T_2 \geqslant 0$.

Let $\varepsilon > 0$. Let $X_{1,2} \in M_+$ be elements with finite spectrum so that $X_1 \geqslant T_1$, $X_2 \leqslant T_2$, $\|X_1 - T_1\| < \varepsilon/2$, $\|X_2 - T_2\| < \varepsilon/2$. Since $|||T_1q|||_{\text{ess}} \geqslant |||T_2q|||_{\text{ess}}$ for all $q \leqslant p$, we also have $|||X_1q|||_{\text{ess}} \geqslant |||X_2q|||_{\text{ess}}$ for all $q \leqslant p$. Moreover, by subtracting a compact operator from each X_j, if necessary, we may assume $X_j = \Sigma_i c_i^j f_i^j$, where $c_i^j > 0$ and f_i^j are properly infinite, mutually orthogonal projections for all i, j. Assume in addition that all f_i^j have the same central support. Then by 2.6.1 we have $|||(X_1 + X_2)p|||_{\text{ess}} = |||X_1p|||_{\text{ess}}$ and since $|||f_1Tf_1p|||_{\text{ess}} + \varepsilon/2 \geqslant |||X_1p|||_{\text{ess}} = |||(X_1 + X_2)p|||_{\text{ess}} \geqslant |||(f_1Tf_1 + f_2Tf_2)p|||_{\text{ess}} - \varepsilon/2$, tending with ε to zero, we get the result (the reverse inequality is trivial) in the case where all f_i^j have the same central support. Now the general case reduces immediately to this one by 2.6.3. Q.E.D.

2.8. Since the norm $|||\ |||$ is a supremum of vector norms, it is inferior semicontinuous with respect to the weak operator topology. Indeed, if T_i tends in the weak operator topology to T, then $\|T\xi\| \leqslant \lim_i \sup \|T_i \xi\|$, so that

$$|||T||| = \sup\left\{ \|Tx\|_\varphi | x \in M^1_{\varphi,\psi} \right\} \leqslant \lim_i \sup\left(\sup\left\{ \|T_i x\|_\varphi | x \in M^1_{\varphi,\psi} \right\}\right.$$
$$= \lim_i \sup |||T_i|||.$$

2.9. We now prove a version of Johnson and Parrott's trick in [3].

LEMMA. *Let $N \subset M$ be a von Neumann algebra and $T \in M$ such that $(\operatorname{ad} T)(N) \subset \mathscr{J}(M)$ and $T \notin \mathscr{J}(M)$. Suppose the set $\mathscr{P} = \{ f \in \mathscr{P}(N) |\ |||fTf|||_{\mathrm{ess}} = |||T|||_{ess} \}$ contains no minimal projections. Then there exist a $c > 0$ and a sequence of mutually orthogonal projections $\{ e_n \}_n$ in N such that $|||e_n T e_n||| \geqslant c$ for all n.*

Proof. Let $\mathscr{F}$ be a maximal chain in $\mathscr{P}$ and let $f_0 = \inf \mathscr{F}$. Since $\mathscr{P}$ has no minimal projections, $f_0 \notin \mathscr{P}$. Thus $c = (|||T|||_{\mathrm{ess}} - |||f_0 T f_0|||_{\mathrm{ess}})/2 > 0$. Then the chain $\mathscr{F}' = \{ f - f_0 | f \in \mathscr{F} \}$ decreases to zero, and since

$$|||(f - f_0)T(f - f_0)|||_{\mathrm{ess}} + |||f_0 T f_0|||_{\mathrm{ess}} \geqslant |||(f - f_0)T(f - f_0) + f_0 T f_0|||_{\mathrm{ess}}$$
$$= |||fTf|||_{\mathrm{ess}} = |||T|||_{\mathrm{ess}},$$

it follows that $|||f'Tf'|||_{\mathrm{ess}} \geqslant 2c$ for any $f' \in \mathscr{F}'$.

We can now construct recursively the required sequence $\{ e_n \}_{n \in \mathbb{N}}$. Assume $f_1', \ldots, f_n'$ are n projections in $\mathscr{F}'$ with $|||(f_k' - f_{k-1}')T(f_k' - f_{k-1}')||| \geqslant c$, $n \geqslant k > 1$. Since $\mathscr{F}'$ is a chain decreasing to zero, by the inferior semicontinuity of the norm $|||\ |||$ it follows that there exists a projection $f_{n+1}' \in \mathscr{F}'$ with $f_{n+1}' \leqslant f_n'$ such that

$$|||(f_n' - f_{n+1}')T(f_n' - f_{n+1}')||| \geqslant |||f_n' T f_n'|||/2.$$

Thus $|||f_n' T f_n'||| \geqslant |||f_n' T f_n'|||_{\mathrm{ess}} \geqslant 2c$, and consequently

$$|||(f_n' - f_{n+1}')T(f_n' - f_{n+1}')||| \geqslant c,$$

so that $e_n = f_{n+1}' - f_n'$ will do. Q.E.D.

2.10. Let now M be an arbitrary semifinite von Neumann algebra and $N \subset M$ a weakly closed *-subalgebra of it. Let δ: $N \mapsto \mathscr{J}(M)$ be a derivation. By [3] δ is norm continuous and by [2] it is weakly continuous. Let p be the unit of N and $K = \delta(p)p - p\delta(p) \in \mathscr{J}(M)$. Then $Kp - pK = \delta(p)p - 2p\delta(p)p + p\delta(p) = (\delta(p) - p\delta(p)) - (2\delta(p^2)p - 2\delta(p)p^2) + p\delta(p) = \delta(p)$ so that $(\delta - \operatorname{ad} K)(p) = 0$ and $(\delta - \operatorname{ad} K)(x) = (\delta - \operatorname{ad} K)(pxp) = p(\delta - \operatorname{ad} K)(x)p$, which shows that $\delta - \operatorname{ad} K$ takes values in pMp.

This shows that in order to prove Theorem 1.1, we may assume the weakly closed *-subalgebra $N \subset M$ has the same unit as M, i.e., N is a von Neumann subalgebra of M. Therefore, in all the rest of the paper the subalgebra N will be considered to have the same unit as M.

2.11. Let $\{p_i\}_{i\in I}$ be a family of mutually orthogonal projections in the center of M with $\Sigma_i p_i = 1$. Assume that for each i there exists $K_i \in \mathscr{J}(M)p_i = \mathscr{J}(M_{p_i})$ such that $\delta(x)p_i = \operatorname{ad} K_i(x)$ and $\|K_i\| \leqslant 2\|\delta\|$ for all $x \in N$. Then $K = \Sigma_{i\in I}K_i$ is in $\mathscr{J}(M)$ and $\delta = \operatorname{ad} K$ on N.

Since in a semifinite von Neumann algebra there exist mutually orthogonal central projections p_i with $\Sigma p_i = 1$ such that each $\mathscr{Z}(M)p_i$ is countable decomposable (or, equivalently, has a normal faithful state), it follows by the above observation that to prove Theorem 1.1 for general M it is sufficient to prove it for each M_{p_i}, i.e., under the assumption that $\mathscr{Z}(M)$ is of countable type. Thus we may and will assume in the rest of the paper that M has countable decomposable center $\mathscr{Z}(M)$, that ψ is a normal faithful state on $\mathscr{Z}(M)$, and that φ is the unique normal faithful trace on M associated to ψ as in 2.1. The reader will note that each time we get a $K \in \mathscr{J}(M)$, $\delta = \operatorname{ad} K$ for M of countable type, we also have $\|K\| \leqslant 2\|\delta\|$.

2.12. Let $N_0 \subset N$ be a finite-dimensional von Neumann subalgebra of N, $\mathscr{U}_0$ the unitary compact group of N_0, and λ the normalized Haar measure on $\mathscr{U}_0$. Then $K = \int\delta(u)u^*\,d\lambda(u) \in \mathscr{J}(M)$ satisfies for any $u_0 \in \mathscr{U}_0$:

$$\begin{aligned} Ku_0 - u_0K &= \int\delta(u)u^*u_0\,d\lambda(u) - \int u_0\delta(u)u^*\,d\lambda(u) \\ &= \int\delta(u)(u_0^*u)^*\,d\lambda(u) - \int u_0\delta(u)u^*\,d\lambda(u) \\ &= \int\delta(u_0u)u^*\,d\lambda(u) - \int u_0\delta(u)u^*\,d\lambda(u) \\ &= \delta(u_0)\int d\lambda(u) + \int u_0\delta(u)u^*\,d\lambda(u) - \int u_0\delta(u)u^*\,d\lambda(u) \\ &= \delta(u_0). \end{aligned}$$

Thus $(\delta - \operatorname{ad} K)(x_0) = 0$ for any $x_0 \in N_0$. In particular, this shows that if N is a finite direct sum, then to prove 1.1 for $N \subset M$ it is sufficient to prove it for each summand.

3. The atomic abelian case. In this section we prove Theorem 1.1 in the case N is isomorphic to the algebra $\ell^\infty(I)$ for a set I of arbitrary cardinality.

To do this, let $\{e_i\}_{i\in I}$ be the minimal projections of $N = \ell^\infty(I)$ and note first that the series $\Sigma_{i\in I}\delta(e_i)e_i$ is convergent in the strong operator topology. Indeed, the sequence is bounded because if $e_1, e_2, \dots, e_n \in \{e_i\}_{i \in I}$, then

$$(*) \qquad \sum_{k=1}^{n} \delta(e_k)e_k = \sum_{k,l=1}^{n} \int z_k \bar{z}_l \delta(e_k) e_l \, d\lambda(z)$$

$$= \int \delta\left(\sum_{k=1}^{n} z_k e_k\right)\left(\sum_{l=1}^{n} \bar{z}_l e_l\right) d\lambda(z),$$

where λ is the normalized Haar measure on the torus $\mathbb{T}^n$ and $z = (z_1, z_2, \dots, z_n) \in \mathbb{T}^n$, so that

$$\left\| \sum_{k=1}^{n} \delta(e_k)e_k \right\| \leqslant \int \left\| \delta\left(\sum_{k=1}^{n} z_k e_k\right)\left(\sum_{l=1}^{n} \bar{z}_l e_l\right) \right\| d\lambda(z) \leqslant \|\delta\|.$$

Now if M is normally represented on some Hilbert space $\mathscr{H}$, $\xi \in \mathscr{H}$, and $\varepsilon > 0$, then there exists a finite set $I_0 \subset I$ such that $\|\xi - (\sum_{i \in I_0} e_i)\xi\| < \varepsilon$, and thus for any finite set $J_0 \subset I$ with $J_0 \cap I_0 = \varnothing$ we have

$$\left\| \sum_{i \in J_0} \delta(e_j)e_j\xi \right\| \leqslant \varepsilon\|\delta\| + \left\| \left(\sum_{j \in J_0} \delta(e_j)e_j\right)\left(\sum_{i \in I_0} e_i\right)\xi \right\| = \varepsilon\|\delta\|,$$

which shows that $\sum_{i \in I} \delta(e_i)e_i\xi$ is convergent for any $\xi \in \mathscr{H}$.

Let $T = \sum_{i \in I} \delta(e_i)e_i$. Since δ is a derivation and $(\sum_{i \in I} \delta(e_i)e_i)e_{i_0} = \delta(e_{i_0})e_{i_0}$, we have

$$Te_{i_0} - e_{i_0}T = \delta(e_{i_0})e_{i_0} - \sum_{i \in I} e_{i_0}\delta(e_i)e_i$$

$$= \delta(e_{i_0})e_{i_0} - \sum_{i \in I} \delta(e_{i_0}e_i)e_i + \delta(e_{i_0}) \sum_{i \in I} e_i$$

$$= \delta(e_{i_0})e_{i_0} - \delta(e_{i_0})e_{i_0} + \delta(e_{i_0}) = \delta(e_{i_0}).$$

Since both δ and $\operatorname{ad} T$ are weakly continuous on N and the linear span of $\{e_i\}_{i \in I}$ is weakly dense in $N = \ell^\infty(I)$, it follows that $\delta = \operatorname{ad} T$ on N.

We show that T is in $\mathscr{J}(M)$. Suppose $T \notin \mathscr{J}(M)$. Let

$$\mathscr{P} = \{ f \in \mathscr{P}(N) \mid |||fTf|||_{\text{ess}} = |||T|||_{\text{ess}} \}.$$

Then $\mathscr{P}$ contains no minimal projections, because if $e \in \mathscr{P}$ is a minimal projection of $\mathscr{P}$ and $e_0 \leqslant e$ is a minimal projection of N, then $e_0Te_0 = 0$ (by the definition of T), so that $e - e_0 \in \mathscr{P}$, a contradiction. Thus by 2.9 there exist

$c > 0$ and a sequence of mutually orthogonal projections $\{f_n\}_{n \in \mathbb{N}}$ in N such that

$$|||f_n T f_n||| \geqslant c \quad \text{for all } n.$$

Moreover, by the inferior semicontinuity of the norm $|||\ |||$ we may assume each projection f_n is the sum of a finite set $J_n \subset J$ of minimal projections in N. But by $(*)$ we have

$$Tf_n = \sum_{j \in J_n} \delta(e_j)e_j = \int \delta\Big(\sum_{i \in J_n} z_i e_i\Big)\Big(\sum_{j \in J_n} \bar{z}_j e_j\Big)\, d\lambda(z),$$

so that

$$\int \Big|\Big|\Big| f_n \delta\Big(\sum_{i \in J_n} z_i e_i\Big)\Big(\sum_{j \in J_n} \bar{z}_j e_j\Big) f_n \Big|\Big|\Big|\, d\lambda(z) \geqslant |||f_n T f_n||| \geqslant c,$$

which implies that for some $u_n = \sum_{i \in J_n} z_i e_i$,

$$|||f_n \delta(u_n) u_n^* f_n||| \geqslant c.$$

Now let $u = \sum_{n \in \mathbb{N}} u_n$. Then, for each n,

$$f_n \delta(u) u^* f_n = f_n \delta(f_n u) u_n^* f_n - f_n \delta(f_n) f_n = f_n \delta(u_n^*) u_n f_n,$$

so that

$$|||f_n \delta(u) u^* f_n||| = |||f_n \delta(u_n^*) u_n f_n||| \geqslant c.$$

Since $\delta(u)u^*$ is in $\mathscr{J}(M)$, by 2.5 this is a contradiction. Thus $\sum_{i \in I} \delta(e_i)e_i$ is in $\mathscr{J}(M)$, and the case $N = \ell^\infty(I)$ is solved.

4. The continuity result. For the next result we assume $N \subset M$ is a finite von Neumann algebra with a normal faithful finite trace τ, $\tau(1) = 1$. We denote by $\|x\|_2 = \tau(x^* x)^{1/2}$, $x \in N$.

4.1. PROPOSITION. *Let δ: $N \mapsto \mathscr{J}(M)$ be a derivation. Then δ is continuous from the unit ball of N with the strong operator topology into $\mathscr{J}(M)$ with the norm $|||\ |||$.*

Proof. We first prove that if $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of projections in N with $\tau(f_n) \mapsto 0$, then $|||\delta(f_n)||| \mapsto 0$. Suppose $|||\delta(f_n)|||$ does not converge to 0. By taking a subsequence, if necessary, we may assume that $|||\delta(f_n)||| \geqslant c$ for some $c > 0$ and all n and that $\sum \tau(f_n) < \infty$. Let g_n be the supremum of $\{f_k\}_{k \geqslant n}$. Then $\tau(g_n) \leqslant \sum_{k \geqslant n} \tau(f_k)$ tends to zero with n. Denote by $s_{n,m}$ the support of $f_m g_n f_m$. Then $s_{nm} \leqslant f_m$ and $s_{n,m}$ is majorized by g_n and thus, τ being a trace,

$\tau(s_{nm}) \leqslant \tau(g_n) \mapsto_n 0$ for each m. Since $\{g_n\}_{n\in\mathbb{N}}$ is decreasing, $\{f_m g_n f_m\}_{n\in\mathbb{N}}$ is decreasing so that $\{s_{nm}\}_{n\in\mathbb{N}}$ is decreasing for each m. Thus $\{f_m - s_{nm}\}_{n\in\mathbb{N}}$ increases to f_m, so that $\{\delta(f_m - s_{nm})\}_{n\in\mathbb{N}}$ is weakly convergent to $\delta(f_m)$. By the inferior semicontinuity of the norm $|||\ |||$ (cf. 2.8) it follows that for a fixed m, if n is big enough, $|||\delta(f_m - s_{nm})||| \geqslant c/2$.

We may thus get by induction an increasing sequence of integers $n_1, n_2, \ldots$ such that the projections $h_k = f_{n_k} - s_{n_{k+1}, n_k}$ satisfy $|||\delta(h_k)||| \geqslant c/2$. These projections also satisfy $\tau(h_k) \leqslant \tau(f_{n_k}) \mapsto_k 0$.

Moreover, since $h_k \leqslant f_{n_k}$ and s_{n_{k+1}, n_k} is the support of $f_{n_k} g_{n_{k+1}} f_{n_k}$, by the definition of h_k we get

$$h_k g_{n_{k+1}} h_k = h_k f_{n_k} g_{n_{k+1}} f_{n_k} h_k \leqslant h_k s_{n_{k+1}} f_{n_k} h_k \leqslant h_k s_{n_{k+1}, n_k} h_k = 0.$$

Thus $h_k g_{n_{k+1}} = 0$, in particular $h_k f_{n_l}$ for $l \geqslant k+1$, and so $h_k h_l = 0$, which means that h_k are all mutually orthogonal projections. Since we also have $|||\delta(h_k)||| \geqslant c/2$, we obtain a contradiction, by the atomic abelian case (§3) and 2.5.

Now we turn to the general case. Since $\|\ \|_2$ induces the strong operator topology on the unit ball of M, we have to show that if $(x_n)_n$ is a bounded sequence in M with $\|x_n\| \mapsto_2 0$, then $|||\delta(x_n)||| \to 0$. It is clear that we only need to prove this implication in the case where x_n are self-adjoint elements and $\|x_n\| \leqslant 1$. Moreover, since $\|\,|x_n|\,\|_2 = \|x_n\|_2$, it follows that if $\|x_n\|_2 \mapsto 0$, then $\|(x_n)_+\|_2 \mapsto 0$ and $\|(x_n)_-\|_2 \mapsto 0$, so that it is sufficient to prove that if x_n are positive elements and $\|x_n\|_2 \mapsto 0$ (equivalently, $\tau(x_n) \mapsto 0$), then $|||\delta(x_n)||| \mapsto 0$.

Let $x_n = \sum_{m\geqslant 1} 2^{-m} e_m^n$ be the diadic decomposition of x_n. It follows that $\tau(e_m^n) \mapsto_n 0$ for each $m \geqslant 1$. Let $\varepsilon > 0$ and $m_0 \geqslant 1$ so that $2^{-m_0} \leqslant \varepsilon/2$. Then by the first part of the proof there exists n_0 such that for $n \geqslant n_0$, $|||\delta(e_m^n)||| < \varepsilon/2$ for any $m \leqslant m_0$. Thus, for $n \geqslant n_0$ we get

$$|||\delta(x_n)||| \leqslant \sum_{m=1}^{m_0} 2^{-m} |||\delta(e_m^n)||| + \|\delta\| \sum_{m>m_0} 2^{-m} \leqslant \varepsilon. \qquad \text{Q.E.D.}$$

The above continuity result will enable us to reduce the theorem to more tractable situations and to prove it in several cases. We will actually use the following consequence of 4.1.

4.2. Corollary. *Let* $K_\delta = \overline{\mathrm{co}}^{w}\{\delta(u)u^* \mid u$ *unitary element in* $N\}$. *Assume* N *is finite and countable decomposable and denote by* τ *a normal finite faithful trace on it,* $\tau(1) = 1$. *Given* $\beta > 0$, *there exists* $\alpha > 0$ *such that if* $x \in N$, $\|x\| \leqslant 1$, *and* $\|x\|_2 \leqslant \alpha$, *then*

$$|||Tx||| \leqslant \beta \quad \text{and} \quad |||xT||| \leqslant \beta \quad \text{for all } T \in K_\delta.$$

Proof. By the preceding proposition there exists $\alpha > 0$ such that $\|y\| \leqslant 1$, $\|y\|_2 < \alpha$, implies $|||\delta(y)||| < \beta/3$. Since $\delta(u)u^*y = \delta(y) - u\delta(u^*y)$ and $\|u^*y\|_2 = \|y\|_2$, it follows that

$$|||\delta(u)u^*y||| \leqslant |||\delta(y)||| + |||\delta(u^*y)||| < 2\beta/3$$

for any unitary element u in M. By taking convex combinations of $\delta(u)u^*$ and using the fact that the norm $||| \ |||$ is weak inferior semicontinuous, we get $|||Ty||| < \beta$ for all $T \in \mathrm{K}_\delta$. Similarly, $|||yT||| \leqslant \beta$. Q.E.D.

Actually, we will mostly use 4.1 and 4.2 through the next technical results, which show that in many cases, whenever there exists $T \in K_\delta$ (defined as in 4.2) with $\operatorname{ad} T = \delta$, then $T \in \mathscr{J}(M)$.

First we consider the case when N is abelian and locally compatible with $\mathscr{Z}(M)$ (in the sense of 1.1).

4.3. Proposition. *Assume that the von Neumann subalgebra N of M is abelian and that there exist projections $\{e_i\}_{i\in I}$ in N so that $\sum e_i = 1$ and so that for each i we have either $Ne_i \supset \mathscr{Z}(M)e_i$ or $Ne_i \subset \mathscr{Z}(M)e_i$. Moreover, assume that there exist projections $\{p_j\}_{j\in J}$ in N so that $\sum p_j = 1$, N_{p_j} is of countable type for each j and δ vanishes on the set $\{p_j\}_j$.*

If $T \in K_\delta = \overline{\mathrm{co}}^{\mathrm{w}}\{\delta(u)u^* | u \in \mathscr{U}(N)\}$ is so that $\operatorname{ad} T = \delta$, *then $T \in J(M)$.*

Proof. Assume $T \notin \mathscr{J}(M)$, so that $|||T|||_{\mathrm{ess}} > 0$. Let

$$\mathscr{P} = \{e \in \mathscr{P}(N) \mid |||eTe|||_{ess} = |||T|||_{\mathrm{ess}}\}.$$

If $\mathscr{P}$ contains no minimal projections, then by 2.9 there exists a sequence of mutually orthogonal projections $\{e_n\}_n$ in N so that $|||e_nTe_n||| \geqslant c$ for some $c > 0$ and all n.

By the inferior semicontinuity of the norm $||| \ |||$, for each n we can find a projection p_n in the von Neumann algebra generated by $\{p_i\}_{i\in I}$ such that N_{p_n} is countable decomposable and

$$|||e_nTe_np_n||| \geqslant |||e_nTe_n|||/2 = c/2.$$

Let p be the supremum of $\{p_n\}_n$. Then p belongs to $\{p_i\}''_{i\in I}$ (and thus $\delta(p) = 0$), N_p is countable decomposable and clearly

$$|||e_nTe_np||| \geqslant c/2 \quad \text{for all } n.$$

By construction e_np tends strongly to zero in N_p.

If we consider $\delta'\colon N_p \mapsto \mathscr{J}(M_p)$ defined by $\delta'(xp) = \delta(x)p$, then obviously $Tp \in K_{\delta'}$. Thus by 4.2 we have $|||e_nTe_np||| \to 0$, a contradiction.

Assume now that $\mathscr{P}$ has minimal projections and denote one of them by e. Assume first that $ee_i \neq 0$ for some i with $Ne_i \subset \mathscr{Z}(M)e_i$. Denote $f_0 = ee_i$. Then we also have $Nf_0 \subset \mathscr{Z}(M)f_0$. It follows that for any unitary element $u \in \mathscr{U}(N)$ there is a $z_0 \in \mathscr{U}(\mathscr{Z}(M))$ such that $uf_0 = z_0 f_0$, and we have

$$\begin{aligned} f_0\delta(u)u^*f_0 &= \delta(f_0)f_0 - \delta(f_0u)u^*f_0 \\ &= \delta(f_0)f_0 - (Tf_0u - f_0uT)u^*f_0 \\ &= \delta(f_0)f_0 - (Tz_0f_0 - z_0f_0T)z_0^*f_0 \\ &= \delta(f_0)f_0 - (Tf_0 - f_0T)z_0z_0^*f_0 = 0. \end{aligned}$$

Thus, since $T \in \overline{\text{co}}^{\text{w}}\{\delta(u)u^* | u \in \mathscr{U}(N)\}$, we get $f_0Tf_0 = 0$, which implies that $eTe - (e - f_0)T(e - f_0) \in \mathscr{J}(M)$. Thus $e - f_0 \in \mathscr{P}$, contradicting the minimality of e in $\mathscr{P}$.

Now the only case left is when there is a nonempty set $I_0 \subset I$ so that $e_i \leqslant e$ and $Ne_i \supset \mathscr{Z}(M)e_i$ for all $i \in I_0$ and $\sum_{i \in I_0} e_i = e$. Fix an i in I_0 with $e_i \neq 0$. If $Ne_i = \mathscr{Z}(M)e_i$, then the first case applies and leads to a contradiction, so we may assume $Ne_i \neq \mathscr{Z}(M)e_i$, and in fact we may assume there exists no $e_i' \in Ne_i$ so that $Ne_i' = \mathscr{Z}(M)e_i'$. It then follows that there exists a projection $f_i \in Ne_i$ so that $qf_i \notin \mathscr{Z}(M)e_i$ for any $q \in \mathscr{Z}(M)$, $q \neq 0$.

Then by 2.7 it follows that there exists a projection $q_i' \in \mathscr{Z}(M)$ such that

$$|||Tf_iq_i'|||_{\text{ess}} = |||Te_iq_i'|||_{\text{ess}},$$

$$\left|\left|\left|T(e_i - f_i)(1 - q_i')\right|\right|\right|_{\text{ess}} = \left|\left|\left|Te_i(1 - q_i')\right|\right|\right|_{\text{ess}}.$$

Thus if we denote by $e_i' = f_iq_i' + (e_i - f_i)(1 - q_i')$, then we have $e_i' \leqslant e_i$, $e_i' \neq e_i$, $e_i' \in \mathscr{P}(N)$, and by 2.3.2 we have $|||Te_i'|||_{\text{ess}}^2 = |||Tf_iq_i'|||_{\text{ess}}^2 + |||T(e_i - f_i)(1 - q_i')|||_{\text{ess}}^2 = |||Te_iq_i'|||_{\text{ess}}^2 + |||Te_i(1 - q_i')|||_{\text{ess}}^2 = |||Te_i|||_{\text{ess}}^2$.

Denote by $e' = (e - e_i) + e_i'$. Then $e' \in N$, $e' \leqslant e$ and $e' \neq e$.

Then let $q_i \in \mathscr{Z}(M)$ be a projection satisfying

$$|||Te_iq_i|||_{\text{ess}} = |||Teq_i|||_{\text{ess}},$$

$$\left|\left|\left|T(e - e_i)(1 - q_i)\right|\right|\right|_{\text{ess}} = \left|\left|\left|Te(1 - q_i)\right|\right|\right|_{\text{ess}}.$$

Then we have

$$\begin{aligned} |||Te|||_{\text{ess}}^2 &= \left|\left|\left|T(e - e_i)(1 - q_i)\right|\right|\right|_{\text{ess}}^2 + |||Te_iq_i|||_{\text{ess}}^2 \\ &= \left|\left|\left|T(e - e_i)(1 - q_i)\right|\right|\right|_{\text{ess}}^2 + |||Te_i'q_i|||_{\text{ess}}^2 \\ &= \left|\left|\left|T(e_i'q_i + (e - e_i)(1 - q_i))\right|\right|\right|_{\text{ess}}^2 \\ &\leqslant \left|\left|\left|T(e_i' + e - e_i)\right|\right|\right|_{\text{ess}}^2 = |||Te'|||_{\text{ess}}^2 \leqslant |||Te|||_{\text{ess}}^2. \end{aligned}$$

Thus $|||Te|||_{\text{ess}} = |||Te'|||_{\text{ess}}$, which again contradicts the minimality of e. This ends the proof of the proposition. Q.E.D.

4.4. PROPOSITION. *Let $N \subset M$ be a von Neumann subalgebra of type* II_1. *Assume the derivation δ: $N \mapsto \mathscr{J}(M)$ vanishes on a set of projections $\{p_i\}_i \subset \mathscr{Z}(M)$ with the property that $\sum p_i = 1$ and N_{p_i} is countable decomposable for all i. If $T \in K_\delta$ is such that* $\operatorname{ad} T = \delta$ *on N, then $T \in \mathscr{J}(M)$.*

Proof. Since N is of type II_1, there exists a decreasing sequence of projections $\{e_n\}_{n \geqslant 0}$ in N with $e_0 = 1$, $e_{n+1} \sim e_n - e_{n+1}$ for all $n \geqslant 0$. Suppose we have shown that for some $n \geqslant 0$ we have $|||e_k T e_k|||_{\text{ess}} = |||T|||_{\text{ess}}$ for all $k \leqslant n$. Let u_n be a unitary element in N such that $u_n e_{n+1} u_n^* = e_n - e_{n+1}$. Since $u_n T u_n^* - T \in \mathscr{J}(M)$, we have $|||e_{n+1} T e_{n+1} q|||_{\text{ess}} = |||u_n e_{n+1} T e_{n+1} u_n^* q|||_{\text{ess}} = |||u_n e_{n+1} u_n^* T u_n e_{n+1} u_n^* q|||_{\text{ess}} = |||(e_n - e_{n+1}) T (e_n - e_{n+1}) q|||_{\text{ess}}$ for any central projection $q \in \mathscr{Z}(M)$. Since

$$|||e_n T e_n|||_{\text{ess}} = \left|\left|\left| e_{n+1} T e_{n+1} + (e_n - e_{n+1}) T (e_n - e_{n+1}) \right|\right|\right|_{\text{ess}},$$

by 2.7 and 2.3.2 it follows that $|||e_{n+1} T e_{n+1}|||_{\text{ess}} = |||e_n T e_n|||_{\text{ess}}$. Thus $|||e_n T e_n|||_{\text{ess}} = |||T|||_{\text{ess}}$ for all n.

Assume $T \notin \mathscr{J}(M)$, so that $|||e_n T e_n|||_{\text{ess}} \geqslant c > 0$ for all n.

Then the proof continues exactly the same way as the first part of the proof of 4.3 and leads to a contradiction. Q.E.D.

4.5. LEMMA. *Let $A \subset N$ be an abelian von Neumann subalgebra of N and suppose there exists a decreasing sequence of nonzero projections $\{e_n\}_n$ in A such that $e_n \downarrow 0$, $e_0 = 1$, and e_{n+1} is equivalent to $e_n - e_{n+1}$ in N for all $n \geqslant 0$. If $T \in \mathrm{K}_{\delta, A} = \overline{\mathrm{co}}^{\mathrm{w}}\{\delta(u)u^* | u$ unitary element in $A\}$ is so that* $\operatorname{ad} T = \delta$ *on N, then $T \in \mathscr{J}(M)$.*

Proof. If $v_n \in N$ are so that $v_n^* v_n = e_{n+1}$, $v_n v_n^* = e_n - e_{n+1}$, then we have $v_n T - T v_n \in \mathscr{J}(M)$ and for any $q \in \mathscr{Z}(M)$,

$$\begin{aligned} |||e_{n+1} T e_{n+1} q|||_{\text{ess}} &= |||v_n e_{n+1} T e_{n+1} v_n^* q|||_{\text{ess}} = |||v_n e_{n+1} v_n^* T v_n e_{n+1} v_n^* q|||_{\text{ess}} \\ &= |||(e_n - e_{n+1}) T (e_n - e_{n+1}) q|||_{\text{ess}}. \end{aligned}$$

The rest of the proof is exactly as the proof of 4.4. Q.E.D.

We end this section by proving a useful converse to the preceding results. Note that the proof doesn't use the continuity result 4.1.

4.6. LEMMA. *Let N be an arbitrary von Neumann subalgebra of M and δ: $N \mapsto \mathscr{J}(M)$ a derivation. If there exists $K \in \mathscr{J}(M)$ such that $\delta =$* $\operatorname{ad} K$, *then there exists $T \in K_\delta$ such that $\delta =$* $\operatorname{ad} T$.

Proof. Assume first that $\varphi(K^*K) < \infty$. Let

$$C = \overline{\mathrm{co}}^{\mathrm{w}}\{uKu^* | u \text{ unitary element in } N\}.$$

Then $\|y\|_\varphi \leqslant \|K\|_\varphi$ for all y in C and C is a weakly compact convex subset of M. By the inferior semicontinuity of the norm $\| \ \|_\varphi$ it follows that there exists a unique element $y_0 \in C$ with $\|y_0\|_\varphi \leqslant \|y\|_\varphi$ for all $y \in C$. Since $uy_0u^* \in C$ and $\|uy_0u^*\|_\varphi = \|y_0\|_\varphi$, it follows that $uy_0u^* = y_0$ for all unitary elements $u \in N$. Thus $y_0 \in C \cap N'$.

Let's now show that for arbitrary K there also exists some $y \in C \cap N'$. To this end note first that for each n there exist (by 2.4) $p_n \in \mathcal{P}(\mathcal{Z}(M))$ and $K_n \in M_\varphi \subset \mathcal{J}(M)$ such that $\|K_n\| \leqslant \|K\|$, $K_n p_n = K_n$, $\|Kp_n - K_n\| \leqslant 1/n$, and $\psi(p_n) \geqslant 1 - 1/n$. Let $C_n = \overline{\text{co}}^{\text{w}}\{uK_nu^* | u \text{ unitary element in } N\}$ and $y_n \in C_n \cap N'$ (cf. the first part of the proof). Then the Hausdorf distance between C_n and p_nC satisfies

$$\mathrm{d}(p_nC, C_n) \leqslant \|Kp_n - K_n\| \leqslant 1/n.$$

Thus there exists $x_n \in Cp_n$ so that $\|x_n - y_n\| \mapsto 0$.

Now let y be a weak limit point of $\{y_n\}_n$. It follows that $y \in N'$ (because $y_n \in N'$ for each n) and $y \in C$ (because y is also a limit point of $\{x_n\}_n$).

Now set $T = K - y$. Then $K - y \in K - C = \overline{\text{co}}^{\text{w}}\{K - uKu^* | u \text{ unitary element in } N\} = \mathrm{K}_\delta$ and, moreover, since $y \in N'$, $\operatorname{ad} T = \operatorname{ad} K = \delta$. Q.E.D.

5. The type I and properly infinite cases. We first prove Theorem 1.1 when N is a finite type I von Neumann algebra locally compatible with $\mathcal{Z}(M)$ (in the sense explained in section 1 and in 4.3). Since N is finite, there exists a partition of the unity $\{p_i\}_{i \in I}$ in the center of N such that N_{p_i} is countable decomposable for each i. By §3 there exists an element $K_0 \in \mathcal{J}(M)$ such that $(\delta - \operatorname{Ad} K_0)(p_i) = 0$ for all i. Thus we may assume that δ vanishes on $\{p_i\}_{i \in I}$. Moreover, by refining $\{p_i\}_i$ if necessary, we may assume N_{p_i} is homogeneous for each i.

The unitary group of N has an amenable subgroup $\mathcal{U}_0$ such that $\mathcal{U}_0'' = \mathcal{Z}(N)$. Let $T_0 = \int_{\mathcal{U}_0} \delta(u)u^* \, \mathrm{d}\mu_0(u)$, where μ_0 is the invariant mean on $\mathcal{U}_0$ and the integral has the usual significance (see e.g., [2]). Then T_0 is in the K_δ set corresponding to $\mathcal{Z}(N)$, and by the same computations as in 2.12 we have

$$T_0u_0 - u_0T_0 = \delta(u_0) \qquad \text{for any } u_0 \in \mathcal{U}_0.$$

Since both δ and $\operatorname{ad} T_0$ are weakly continuous and $\mathcal{Z}(N)$ is the closed linear span of $\mathcal{U}_0$, it follows that $\delta = \operatorname{ad} T_0$ on $\mathcal{Z}(N)$ and thus 4.3 applies to get that $T_0 \in \mathcal{J}(M)$. Thus, by taking $\delta - \operatorname{ad} T_0$ instead of δ if necessary, we may assume δ vanishes on $\mathcal{Z}(N)$.

Since N_{p_i} is homogeneous, we have $N_{p_i} \simeq M_{n(i)} \otimes \mathcal{Z}_i$ for some $n(i) \times n(i)$ matrix algebra $M_{n(i)}$ and an abelian algebra $\mathcal{Z}_i$. Let $\mathcal{U}_i$ be the unitary group of $M_{n(i)} \otimes 1$ and $\mathcal{U} = \oplus \mathcal{U}_i$. Then $\mathcal{U}$ is amenable with an invariant mean μ. Set $T = \int_{\mathcal{U}} \delta(u)u^* \, \mathrm{d}\mu(u)$. Then T is in the K_δ set corresponding to N and the integral $p_i \int_{\mathcal{U}_i} \delta(u)u^* \, \mathrm{d}\mu(u) p_i = p_i \int_{\mathcal{U}} \delta(u)u^* \, \mathrm{d}\mu(u) p_i$ is norm convergent (since $\mathcal{U}_i$ is compact). Thus $p_iTp_i \in \mathcal{J}(M)$ (as a norm limit of elements $\delta(u)u^*$

which are in $\mathscr{J}(M)$). Moreover, ad T equals δ on $\mathscr{U}$ and also on $\mathscr{Z}(N)$. But $\mathscr{U}$ and $\mathscr{Z}(N)$ generate N, so that ad $T = \delta$ on N. Now if $T \notin \mathscr{J}(M)$ and, $\mathscr{P} = \{e \in \mathscr{P}(N) \mid |||eTe|||_{\mathrm{ess}} = |||T|||_{\mathrm{ess}}\}$, then it follows that $\mathscr{P}$ has no minimal projections (if e would be such a minimal projection, then $ep_i \neq 0$ for some i and $e - ep_i$ contradicts the minimality of e). To get a contradiction from this, we continue exactly as in the proofs of 4.3 or 4.5. Thus T lies in $\mathscr{J}(M)$ and the proof of Theorem 1.1 in the case where N is finite type I is completed.

Assume now that N is properly infinite. Then N and M are isomorphic to $N_1 \overline{\otimes} \mathscr{B}(l^2(\mathbb{Z}))$ and $M_1 \otimes \mathscr{B}(l^2(\mathbb{Z}))$, respectively, where $N_1 \subset M_1$ are von Neumann algebras, in such a way that the inclusion $N \subset M$ becomes $N_1 \overline{\otimes} \mathscr{B}(l^2(\mathbb{Z})) \subset M_1 \overline{\otimes} \mathscr{B}(l^2(\mathbb{Z}))$. Note first that if the derivation $\delta\colon N \mapsto \mathscr{J}(M)$ vanishes on $\mathbb{C}1 \overline{\otimes} \mathscr{B}(l^2(\mathbb{Z})) \subset N = N_1 \otimes \mathscr{B}(l^2(\mathbb{Z}))$, then, given a unitary $u \in N_1 \otimes \mathbb{C}1$, we have for any $x \in \mathbb{C}1_{M_1} \otimes \mathscr{B}(l^2(\mathbb{Z}))$,

$$\delta(u)x = \delta(ux) = \delta(xu) = x\delta(u),$$

so that $\delta(u) \in \mathscr{J}(M) \cap (\mathbb{C}1 \otimes \mathscr{B}(l^2(\mathbb{Z}))' \cap M_1 \overline{\otimes} \mathscr{B}(l^2(\mathbb{Z})) = J(M) \cap (M_1 \otimes \mathbb{C}1_{\mathscr{B}(l^2(\mathbb{Z}))}) = 0$. Thus $\delta = 0$ on N.

From this it follows that to prove the properly infinite case it is sufficient to prove the case when $\delta\colon N = \mathscr{B}(l^2(\mathbb{Z})) \mapsto \mathscr{J}(M)$.

Let D be the diagonal von Neumann subalgebra of $\mathscr{B}(l^2(\mathbb{Z}))$ and L the von Neumann algebra generated by the bilateral shift u. Let $\sigma(x) = uxu^*$ for $x \in D$ be the automorphism of D implemented by the shift u. By §3 we may assume δ vanishes on D. Then for any $x \in D$ we have

$$\begin{aligned} x\delta(u^n)u^{-n} &= \delta(xu^n)u^{-n} = \delta(u^n\sigma^{-n}(x))u^{-n} \\ &= \delta(u^n)\sigma^{-n}(x)u^{-n} = \delta(u^n)u^{-n}\sigma^n(\sigma^{-n}(x)) = \delta(u^n)u^{-n}x, \end{aligned}$$

which shows that $\delta(u^n)u^{-n} \in D' \cap M$ for all $n \in \mathbb{Z}$.

But if we take T to be a (weak) mean (after n) of $\delta(u^n)u^{-n}$, then $T \in D' \cap M$ and, as in the preceding proof of the type I case, we have

$$\delta|_L = \operatorname{ad} T|_L.$$

Thus ad T equals δ on both D and L. Since δ and ad T are weakly continuous derivations, it follows that $\delta = \operatorname{ad} T$ on the von Neumann algebra generated by D and L, which is easily seen to be $\mathscr{B}(l^2(\mathbb{Z})) = N$. Since T belongs to the K_δ set corresponding to L, 4.5 applies to get that $T \in \mathscr{J}(M)$.

6. Some technical results. To prove the remaining type II_1 case of the theorem we need some technical devices that we prove below. As before, we continue to assume that M is countable decomposable and use the notations of section 1.

6.1. LEMMA. *Let N be a von Neumann algebra without atoms, ψ a normal faithful state on N and $\{w_n\}_n$ a sequence of unitary elements in N such that $\psi(w_n^k) \mapsto_n 0$ for all $k \neq 0$. Then there exist unitary elements $\{v_n\}_n$ in N such that $\psi(v_n^k) = 0$, $k \neq 0$, and $\|w_n - v_n\| \to 0$.*

Proof. The proof is the same as the proof of 1.3 in [7], but we give it here anyway for the sake of completeness.

Since N has no atoms, each w_n is contained in some diffuse abelian von Neumann subalgebra $A_n \subset N$ with separable predual and $(A_n, \psi|_{A_n})$ can be identified by some measure preserving isomorphism φ_n with $L^\infty(\mathbb{T}, \mu)$, where μ is the normalized Lebesgue measure on the torus $\mathbb{T}$. Moreover, φ_n can be chosen so that $\varphi_n(w_n) = f_n$, where $f_n(e^{2\pi i t}) = e^{2\pi i h_n(t)}$ for some nondecreasing function h_n: $[0,1] \mapsto [0,1]$. By Helly's selection principle there exists a subsequence $\{h_{k_n}\}_n$ tending everywhere to some nondecreasing function h: $[0,1] \to [0,1]$. Thus, if $f(e^{2\pi i t}) = e^{2\pi i h(t)}$, then $\{f_{k_n}\}_n$ tends everywhere to f, so that by Lebesgue's theorem $\int f_{k_n}^p \,d\mu \mapsto \int f^p \,d\mu$ for all p, which by the hypothesis implies $\int f^p \,d\mu = 0$ for $p \neq 0$. Thus $\int q(f)\,d\mu = \int q \,d\mu$ for Laurent polynomials q so that $\int g \circ f \,d\mu = \int g \,d\mu$ for any $g \in L^\infty(\mathbb{T}, \mu)$. In particular, if we define $g_z(e^{2\pi i s}) = \begin{cases} 1 \text{ if } 0 \leqslant s < t \\ 0 \text{ if } t \leqslant s < 1 \end{cases}$, where $z = e^{2\pi i t}$, then we get $\int_{h(s) \leqslant t} d\lambda(s) = \int g_z \circ f \,d\mu = \int g_z \,d\mu = t$, λ being the Lebesgue measure on $[0,1]$. This implies $h(t) = t$ and hence $f(z) = z$ is the identity function on $\mathbb{T}$. Now, since h_{k_n} are monotone and converge everywhere to a continuous function, it follows that h_{k_n} converge uniformly to h, so that $\|f_{k_n} - f\| \to 0$. Since any limit point of f_{k_n} was shown to be equal to the identity f, it follows that $\|f_n - f\| \to 0$.

We can now take $v_n = \varphi_n^{-1}(f)$. Since $\int f^p \,d\mu = 0$, $\psi(v_n^p) = 0$ for all $p \neq 0$. Moreover, $\|w_n - v_n\| = \|\varphi_n(w_n) - \varphi_n(v_n)\| = \|f - f_n\| \to 0$. Q.E.D.

6.2. LEMMA. (1). *Let $N \subset M$ be a von Neumann subalgebra such that $N' \cap M$ contains no finite projections of M. Let $\varepsilon > 0$ and e, f two finite projections of M with $\varphi(e) < \infty, \varphi(f) < \infty$. Then there exists a unitary element $u \in N$ such that $\|fue\|_\varphi < \varepsilon$. Moreover, if N is abelian, then, given any $n \geqslant 1$, there exists a unitary element $u \in N$ such that $\|fu^k e\|_\varphi < \varepsilon$ for $k \neq 0$, $|k| \leqslant n$.*

(2). *If N is of type II_1 and countable decomposable, M is countable decomposable, and $N' \cap M$ contains no finite projections of M, then there exist an approximately finite-dimensional type II_1 von Neumann algebra $R \subset N$ which contains a diffuse abelian von Neumann subalgebra $A \subset R$ such that $A' \cap M$ contains no finite projections of M. Moreover, if N has separable predual, then we can make the construction so that, in addition to the above properties, A is maximal abelian in N.*

Proof. (1). Let φ_n be the semifinite faithful trace on M^{2n} given by $\varphi_n((x_k)_{|k| \leqslant n, k \neq 0}) = \Sigma\varphi(x_k)$. Set $\mathrm{K}_e^n = \overline{\mathrm{co}}^{\mathrm{w}}\{(u^k e u^{-k})_{|k| \leqslant n, k \neq 0} | \, u$ unitary element of $N\} \subset M^{2n}$. Then $\varphi_n(\bar{x}) \leqslant 2n\varphi(e)$ and $\|\bar{x}\|\varphi_n \leqslant 2n\|e\|_\varphi$ for any $\bar{x} \in \mathrm{K}_e^n$.

By the inferior semicontinuity of the norm $\| \quad \|_{\varphi_n}$, there exists a unique element $\bar{x}_0 \in \mathrm{K}_e^n$ with $\|\bar{x}_0\|_{\varphi_n} \leqslant \|\bar{x}\|_{\varphi_n}$ for all $x \in \mathrm{K}_e^n$. But if N is abelian, then for any unitary element $u \in N$, if $\tilde{u} = (u^k)_{|k| \leqslant n, k \neq 0}$, then $\tilde{u}\mathrm{K}_e^n\tilde{u}^* \subset \mathrm{K}_e^n$ and $\|\tilde{u}\bar{x}_0\tilde{u}^*\|_{\varphi_n} = \|x_0\|_{\varphi_n}$ so that, by the uniqueness of $\bar{x}_0$, $\tilde{u}\bar{x}_0\tilde{u}^* = \bar{x}_0$. Thus if $\bar{x}_0 = (x_k)_{|k| \leqslant n, k \neq 0} \neq 0$, then $x_k \neq 0$ for some k and $u^k x_k = x_k u^k$ for any unitary element $u \in N$. Since in a von Neumann algebra N any unitary element $v \in N$ can be written as u^k for some $u \in N$, it follows that $vx_k = x_k v$ for unitary elements $v \in N$, and by taking linear combinations, $yx_k = x_k y$ for all $y \in N$. But $0 < \|x_k\|_\varphi \leqslant \|e\|_\varphi$ and $x_k \in N' \cap M$, a contradiction. This shows that $0 = x_0 \in \mathrm{K}_e^n$, so that given any $\varepsilon > 0$ and any $f \in (M_\varphi)_+$ there is a $u \in \mathscr{U}(N)$ such that $\Sigma_{0 < |k| \leqslant n}\varphi(fu^k e u^{-k}) < \varepsilon^2$. Thus $\|fu^k e\|_\varphi < \varepsilon$ for all $k \neq 0$, $|k| \leqslant n$. If N is arbitrary, we take M instead of M^{2n} and the proof is the same.

(2). The argument we use is similar to the one in [6]. We first prove that if $p \in N$, then $N_p' \cap M_p$ contains no finite projections of M_p. To show this, let $f \neq 0$ be a projection in $N_p' \cap M_p$ and z a projection in the center of N. Then $zf \in N_p' \cap M_p$, and if f is finite in M_p, then zf is finite in M_{zp}. Take z to be so that $fz \neq 0$ and pz divides z, say n times. It follows that the inclusion $N_z \subset M_z$ is the same as $N_{zp} \otimes M_{n \times n} \subset M_{zp} \otimes M_{n \times n}$ and that $f' = zf \otimes I_n \in (N_{zp} \otimes M_{n\times n})' \cap (M_{zp} \otimes M_{n \times n})$. Hence $f' \in N_z' \cap M_z = z(N' \cap M)z \subset N' \cap M$, and if f is finite, then f' is finite, contradicting the hypothesis.

Since M is countable decomposable, there exists an increasing sequence of finite projections $\{f_n\}_n$ in M with $f_n \uparrow 1$. Moreover, by cutting each f_n with a central projection if necessary, we may assume $\varphi(f_n) < \infty$, $n \in \mathbb{N}$.

We now recursively construct an increasing sequence of finite-dimensional von Neumann subalgebras R_k in N with matrix units $\{e_{ij}^{kp}\}_{\substack{1 \leqslant i, j \leqslant n(k,p) \\ 1 \leqslant p \leqslant m(k)}}$ satisfying the following properties:

1. Each $e_{st}^{k-1, r}$ is the sum of some e_{ij}^{kp}.
2. If A_k is the diagonal algebra of R_k generated by $\{e_{ii}^{kp}\}_{i,p}$, then $\|E_{A_k' \cap M}(f_k)\|_\varphi^2 < (3/4)^k$.
3. $n(k, p) \geqslant k$ for each $p = 1, 2, \ldots, m(k)$.

Assume we have constructed these objects up to some k. By (1) it follows that for each $g = e_{11}^{kp}$ there exists a unitary element $u \in N_g$ such that if e is the support of $\Sigma_i e_{1i}^{kp} f_{k+1} e_{i1}^{kp}$, then for each nonzero $x_i = e_{1i}^{kp} f_{k+1} e_{i1}^{kp}$ we have $\varphi(eueu^*) = \|eue\|_\varphi^2 < 1/2\|x_i\|_\varphi^2$. Approximating u in the uniform norm, we may assume it has finite spectrum so that $u = \Sigma\lambda_s e_s$ with $\Sigma e_s = g$ and $|\lambda_s| = 1$. Then, since $\varphi(x_i u x_i u^*) \leqslant \varphi(eueu^*)$, we have

$$\|x_i\|_\varphi^2 = 2\|x_i\|_\varphi^2 - \|x_i\|_\varphi^2 \leqslant \|x_i\|_\varphi^2 + \|ux_iu^*\|_\varphi^2 - 2\varphi(x_iux_iu^*)$$

$$= \|x_i - ux_iu^*\|_\varphi^2 = \left\| \sum_{r \neq s} (\lambda_r \bar{\lambda}_s - 1) e_r x_i e_s \right\|_\varphi^2$$

$$\leqslant 4 \sum_{r \neq s} \|e_r x_i e_s\|_\varphi^2 = 4\|x_i\|_\varphi^2 - 4\sum_r \|e_r x_i e_r\|_\varphi^2.$$

Thus $\Sigma_r \|e_r x_i e_r\|_\varphi^2 \leqslant 3/4\|x_i\|_\varphi^2$. Now we can apply the same trick to $e_r x_i e_r$ instead of x_i and get a refinement $\{e_s^2\}_s$ of the projections $e_r^1 = e_r$, so that $\|\Sigma_s e_s^2 x_i e_s^2\|_\varphi^2 \leqslant (3/4)^2 \Sigma \|e_r^1 x_i e_r^1\|_\varphi^2 \leqslant (3/4)^2 \|x_i\|_\varphi^2$. More generally, we apply the trick $k+1$ times to get projections $g_l = e_l^{k+1}$ so that $\|\Sigma_l g_l x_i g_l\|_\varphi^2 \leqslant (3/4)^{k+1} \|x_i\|_\varphi^2$ and $\Sigma_l g_l = e_{11}^{kp}$.

Now since N is type II_1, each g_l can be divided into $k+1$ mutually orthogonal equivalent projections. Thus we may consider matrix units $\{g_{ab}^l\}_{1 \leqslant a, b \leqslant k+1}$ with $\Sigma_a g_{aa}^l = g_l$. Then easy computations show that if we denote by $\{e_{st}^{k+1, r}\}_{s,t,r}$ an appropriate relabeling of $\{e_{i1}^{kp} g_{ab}^l e_{1j}^{kp}\}_{a,b,l,i,j,p}$, then this matrix unit and the von Neumann algebra R_{k+1} generated by it, together with its diagonal A_{k+1}, will satisfy conditions 1, 2, and 3.

Let $R = \overline{\bigcup_k R_k}^w$. Then condition 3 implies that R is of type II_1.

Let $A = \overline{\bigcup_n A_n}^w$. Suppose $e \in A' \cap M$, $e \neq 0$, is a finite projection of M. Then by cutting e with a projection in $\mathscr{Z}(M)$ if necessary, we may assume $\varphi(e) < \infty$. Since $f_n \uparrow 1$, there exists n such that $\|f_n e f_n - e\|_\varphi < 1/2\|e\|_\varphi$. By the construction of $A_n \subset A$ there exists a partition of the unity $e_1, \dots, e_m$ with projections in A such that

$$\Big\|\sum_i e_i f_n e_i\Big\|_\varphi < 1/2\|e\|_\varphi .$$

But then

$$\Big\|\sum_i e_i f_n e f_n e_i\Big\|_\varphi \leqslant \Big\|\sum_i e_i f_n e_i\Big\|_\varphi < 1/2\|e\|_\varphi ,$$

so that, since $e = \Sigma_i e_i e e_i$,

$$\|e\|_\varphi = \Big\|\sum_i e_i e e_i\Big\|_\varphi < \Big\|\sum_i e_i(f_n e f_n - e)e_i\Big\|_\varphi + \Big\|\sum_i e_i f_n e f_n e_i\Big\|_\varphi < \|e\|_\varphi ,$$

which is a contradiction.

Finally, if we assume N is separable, then it has a normal faithful trace τ and there is a set of elements $\{y_k\}_k \subset N$ dense in N in the norm $\|y\|_2 = \tau(y^* y)^{1/2}$ and we may construct recursively A_k, R_k, so that, in addition to conditions 1, 2, and 3, to satisfy condition 4, $\|E_{A_k' \cap M}(y_i) - E_{A_k}(y_i)\|_2 < 1/k$, $1 \leqslant i \leqslant k$.

Then by [6] it follows that, besides the above properties, A is also maximal abelian in N. Q.E.D.

In the rest of this section $N \subset M$ will be a type II_1 von Neumann subalgebra with a fixed normal finite faithful trace τ, $\tau(1) = 1$. The norm on N given by τ is denoted $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M$. If $B \subset N$ is a von Neumann subalgebra, then E_B denotes the unique normal τ-preserving conditional expectation onto B (cf. [10]).

6.3. LEMMA. *Assume $A \subset N$ is a maximal abelian von Neumann subalgebra of N such that $A' \cap M$ contains no finite projections of M. Let $\varepsilon > 0$, $n \geqslant 1$, e and f finite projections in M_φ and v a unitary element in N. Then there exists a unitary element $u \in A$ such that $\|f(uv)^k e\|_\varphi^2 \leqslant \varepsilon$ for any $k \neq 0$, $|k| \leqslant n$.*

Proof. Since $\varphi(e)$, $\varphi(f) < \infty$, it follows that $\varphi(e \vee f) < \infty$. Since $\|(e \vee f)(uv)^k(e \vee f)\|_\varphi \geqslant \|f(uv)^k e\|_\varphi$, it is sufficient to prove the statement when $e = f$. Since $\|e(uv)^k e\|_\varphi = \|e(uv)^{-k} e\|_\varphi$, we only need to prove the estimates for $k > 0$. We'll actually prove the following more general result:

$(*)$ If $\varepsilon > 0$, $n \geqslant 1$, $\mathscr{F} \subset N$ is a finite self-adjoint set of norm one elements containing the identity and e, f are finite projections in M_φ, then there exists a unitary element $u \in A$ such that

$$\left\| f x_0 \prod_{i=1}^{k} (u x_i) e \right\|_\varphi^2 < \varepsilon$$

for any $1 \leqslant k \leqslant n$ and $x_0, x_1, \ldots, x_k \in \mathscr{F}$.

We first prove $(*)$ in the case $\varphi(xe) \leqslant c\tau(x)$, $\varphi(fx) \leqslant c\tau(x)$, $x \in N_+$, for some constant $c > 0$. Let $\mathscr{W} = \{w$ partial isometry in $A \mid \|f x_0 \prod_{i=1}^k (w x_i) e\|_\varphi^2 \leqslant \varepsilon\tau(w^*w)$ for any $1 \leqslant k \leqslant n$, $x_0, x_1, \ldots, x_k \in \mathscr{F}\}$ and consider on $\mathscr{W}$ the usual order: $w_0 \leqslant w_1$ if w_0 is a restriction of w_1, i.e., $w_0 = w_1 w_0^* w_0$. The set is clearly inductively ordered. Let u be a maximal element of it and suppose $u^*u \neq 1$. Denote by $A_0 = (1 - u^*u)A(1 - u^*u)$, $N_0 = (1 - u^*u)N(1 - u^*u)$, and $\mathscr{F}_0 = \{(1 - u^*u)x_0(\prod_{i=1}^k (u x_i))(1 - u^*u) \mid 1 \leqslant k \leqslant n, x_0, x_1, \ldots, x_n \in \mathscr{F}\}$. By 1.2 in [6], given any $\delta > 0$, there exists a partition of the unity $e_1, \ldots, e_m$ in A_0 such that $\sum_i \|e_i y e_i - E_{A_0}(y) e_i\|_2^2 = \|\sum_i e_i y e_i - E_A(y)\|_2^2 < \delta\tau(1 - u^*u) = \delta \sum_i \tau(e_i)$ for all $y \in \mathscr{F}_0$. It follows that for some $e_0 = e_i$ we have

$$(**) \qquad \|e_0 y e_0 - E_{A_0}(y) e_0\|_2^2 < \delta\tau(e_0), \qquad y \in \mathscr{F}_0.$$

Let $n \geqslant r, s \geqslant 0$, $x \in \mathscr{F}$, y, $y_1, \ldots, y_s \in \mathscr{F}_0$, $x' \in \mathscr{F}^*$, y', $y_1', \ldots, y_r' \in \mathscr{F}_0$ and $w \in A_0 e_0$, $\|w\| \leqslant 1$ and put $\alpha = |\varphi(e x' \prod_{i=1}^r (y_i' w^*) y' f y \prod_{j=1}^s (w y_j) x e)|$, with the convention that a product over a void set equals 1.

If $s = 1$, then by the Cauchy-Schwartz inequality we have

$$\alpha \leqslant \|f y w y_1 x e\|_\varphi \left\| e x' \left(\prod_{i=1}^r y_i' w^* \right) y' f \right\|_\varphi \leqslant \|f y w v_1 x e\|_\varphi \|e\|_\varphi$$

$$\leqslant \|\bar{f} w \bar{e}\|_\varphi \|e\|_\varphi,$$

where $\bar{e}$ is the supremum of the left supports of all the elements of the form zy_1xe, with $x \in \mathscr{F}$, $y_1 \in \mathscr{F}_0$, and $z \in \mathscr{F}_1 = \{\prod_{i=1}^k E_A(y_i)e_0 \mid 0 \leqslant k \leqslant n, y_1, \dots, y_k \in \mathscr{F}_0\}$, and $\bar{f}$ is the supremum of the right supports of all the elements fy with $y \in \mathscr{F}_0$.

If $s \geqslant 2$, then we have

$$\left\| \bar{f} \prod_{i=1}^{s} (wy_i)xe \right\|_\varphi \leqslant \sum_{j=1}^{s-1} \left\| \prod_{i=1}^{j-1} (wy_i)w\big(E_A(y_j)e_0 - e_0y_je_0\big) w^{s-j}\left(\prod_{t=j+1}^{s-1} E_A(y_t) \right) y_sxe \right\|_\varphi + \left\| \bar{f}w^s \prod_{j=1}^{s-1} E_A(y_j)y_sxe \right\|_\varphi$$

$$\leqslant \sum \Big\{ \|(E_A(y_0)e_0 - e_0y_0e_0)w^jzyxe\|_\varphi \,|\, 1 \leqslant j \leqslant s, x \in \mathscr{F}, y_0, y \in \mathscr{F}_0, z \in \mathscr{F}_1 \Big\} + \|\bar{f}w^s\bar{e}\|_\varphi,$$

where $\bar{e}$, $\bar{f}$ are as before. Thus if β denotes the sum in the right-hand side of the above last inequality, then by (* *) we get $\beta \leqslant sNN_0^2N_1c^{1/2}\delta^{1/2}\|e_0\|_2$, where N, N_0, and N_1 are the number of elements in $\mathscr{F}$, $\mathscr{F}_0$, and $\mathscr{F}_1$, respectively.

Thus, by the Cauchy-Schwartz inequality we obtain

$$\alpha \leqslant \left\| ex' \prod_{i=1}^{r} (y_i'w^*)y'fye_0 \right\|_\varphi \big(\beta + \|\bar{f}w^s\bar{e}\|_\varphi\big)$$

$$\leqslant \|fye_0\|_\varphi\big(\beta + \|\bar{f}w^s\bar{e}\|_\varphi\big) \leqslant c^{1/2}\|e_0\|_2\big(\beta + \|\bar{f}w^s\bar{e}\|_\varphi\big)$$

$$\leqslant sNN_0^2N_1c\delta^{1/2}\|e_0\|_2^2 + c^{1/2}\|e_0\|_2\|\bar{f}w^s\bar{e}\|_\varphi.$$

Thus, if δ is such that $snNN_0^2N_1c\delta^{1/2} < \varepsilon 2^{-2n-1}$ and if, using 6.2, we choose w to be a unitary element in $A_0e_0 = Ae_0 \subset e_0Me_0$ such that $c^{1/2}\|\bar{f}w^s\bar{e}\|_\varphi < \varepsilon 2^{-2n-1}\|e_0\|_2$, then we get $\alpha < 2^{-2n}\varepsilon\tau(e_0)$.

We now show that if w is chosen like this, then $u_0 = u + w$ contradicts the maximality of u. Indeed we have for any $1 \leqslant k \leqslant n$ and $x_0, x_1, \dots, x_k \in \mathscr{F}$:

$$\left\| fx_0\left(\prod_{i=1}^{k} (u + w)x_i \right) e \right\|_\varphi^2 \leqslant \left\| fx_0 \prod_{i=1}^{k} (ux_i)e \right\|_\varphi^2 + \sum \alpha,$$

where the α's appearing in the sum are of the form estimated above and there are $2^k - 1$ terms in that sum. It follows that $\Sigma\alpha \leqslant \varepsilon\tau(e_0)$, so that

$$\left\| fx_0\left(\prod_{i=1}^{k}(u+w)x_i\right)e\right\|_\varphi^2 \leqslant \varepsilon(\tau(u^*u)+\tau(w^*w)) = \varepsilon\tau((u+w)^*(u+w)).$$

This ends the proof of $(*)$ in the case $\varphi(xe) \leqslant c\tau(x), \varphi(fx) \leqslant c\tau(x)$, for $x = N_+$.

To prove the general case, i.e., for arbitrary e, f in M_φ, note that given any $\varepsilon > 0$ there exist finite projections $e', f' \in M_\varphi$ with $\|e - e'\|_\varphi < \varepsilon/3$, $\|f - f'\|_\varphi < \varepsilon/3$, and such that $\varphi(xe') \leqslant c\tau(x), \varphi(f'x) \leqslant c\tau(x)$ for some constant $c > 0$. Indeed, since $\varphi(.e), \varphi(f.)$ are in N_*, there exist $X, Y \in L^1(N, \tau)_+$ such that $\varphi(xe) = \tau(xX), \varphi(fx) = \tau(xY)$, for $x \in N$. Thus if E_n, F_n are the spectral projections of X and Y, respectively, corresponding to the intervals $[0, n]$, then $E_n \uparrow 1$, $F_n \uparrow 1$ and $\varphi(xE_neE_n) = \varphi(E_nxE_ne) = \tau(E_nxE_nX) = \tau(xE_nX) \leqslant n\tau(x)$ and, similarly, $\varphi(F_nfF_nx) \leqslant n\tau(x)$. It follows that $\|E_neE_n - e\|_\varphi \mapsto 0$, $\|F_nfF_n - f\|_\varphi \mapsto 0$, so that if e'_n, f'_n are the spectral projections of E_neE_n and F_nfF_n, respectively, corresponding to the interval $[1/2, \infty)$, then any easy computation shows that $\|e'_n - e\|_\varphi \mapsto 0$, $\|f'_n - f\|_\varphi \mapsto 0$ and $\varphi(xe'_n) \leqslant 2\varphi(xE_neE_n) \leqslant 2n\tau(x), \varphi(f'_nx) \leqslant 2\varphi(F_nfF_nx) \leqslant 2n\tau(x)$ (see, e.g., 1.4 in [8]). Now by the first part of the proof, given $\varepsilon > 0$ and $n \geqslant 1$, there exists a unitary element $u \in A$ such that $\|f'x_0\Pi_{i=1}^k(ux_i)e'\|_\varphi \leqslant \varepsilon/3$ for any $1 \leqslant k \leqslant n$, $x_0, x_1, \ldots, x_k \in \mathscr{F}$. But then

$$\left\| fx_0\prod_{i=1}^{k}(ux_i)e\right\|_\varphi < 2\varepsilon/3 + \left\| f'x_0\prod_{i=1}^{k}(ux_i)e'\right\|_\varphi \leqslant 2\varepsilon/3 + \varepsilon/3 = \varepsilon. \quad \text{Q.E.D.}$$

6.4. COROLLARY. *Let $\varepsilon > 0, n \geqslant 1, e, f$ be two finite projections in M_φ and $v \in N$ a unitary element. There exist a finite projection $e_n \in M$ and a unitary element $w \in N$ such that*

1. $\varphi(e_nw^ke_n) = 0$ *for any* $k \neq 0$;
2. $e_n \leqslant e, \varphi(e - e_n) < \varepsilon$;
3. $\|fw^ke_n\| < \varepsilon$ *for* $k \neq 0$, $|k| \leqslant n$;
4. $\|w - uv\| < \varepsilon$ *for some unitary element* $u \in A$.

Proof. First we prove that given any $\varepsilon' > 0$ there exist unitary elements $u \in A$ and $w' \in N$ and a finite projection $e_n \in M$ such that

$(*)$

(a) $e_n \leqslant e, \varphi(e - e_n) < \varepsilon'$;

(b) $\|fw'^ke_n\| < \varepsilon'$ for $k \neq 0$, $|k| \leqslant n$;

(c) $\|w' - uv\| < \varepsilon'$;

(d) $\varphi(ew'^ke) = 0$ for all $k \neq 0$.

Then it follows by (a) and (d) that $|\varphi(w'^k e_n)| \leqslant \varepsilon'$ for any $k \neq 0$ and thus if ε' is small enough and $\varepsilon' \leqslant \varepsilon/2$, by 6.1 there exists a unitary element $w \in N$ such that $\|w - w'\| \leqslant \varepsilon/2n$ and $\varphi(w^k e_n) = 0$ for any $k \neq 0$. But then $\|fw^k e_n\| \leqslant \|fw'^k e_n\| + n\|w - w'\| < \varepsilon$ for $k \neq 0$, $|k| \leqslant n$, and $\|w - uv\| \leqslant \|w - w'\| + \|w' - uv\| \leqslant \varepsilon/2n + \varepsilon/2 \leqslant \varepsilon$.

Now to prove $(*)$ we let $\varepsilon'' > 0$, $n' \geqslant 1$. By the preceding lemma there exists a unitary element $u \in A$ such that $\|(e \vee f)(uv)^k e\|_\varphi < \varepsilon''$ for $k \neq 0, |k| \leqslant n'$. It follows that $|\varphi(e(uv)^k e)| \leqslant \|e\|_\varphi \|e(uv)^k e\|_\varphi < \varepsilon'' \|e\|_\varphi$, for all $k \neq 0$, $|k| \leqslant n'$, and $\varphi(e(uv)^{-k} f(uv)^k e) = \|f(uv)^k e\|_\varphi^2 < \varepsilon''^2$. If e_k' is the spectral projection of $e(uv)^{-k} f(uv)^k e$ corresponding to the interval $(0, \varepsilon'']$, then $e_k' \leqslant e$, $e_k'(uv)^{-k} f(uv)^k e_k' \leqslant \varepsilon''$ and $e - e_k' \leqslant \varepsilon''^{-1} e(uv)^{-k} f(uv)^k e$ so that $\varphi(e - e_k') \leqslant \varepsilon''^{-1} \varepsilon''^2 = \varepsilon''$. Let $e_n = \wedge\{e_k' | k \neq 0, |k| \leqslant n\}$. Then $e_n \leqslant e$, $\varphi(e_n) \geqslant \varphi(e) - 2n\varepsilon''$, and $\|f(uv)^k e_n\|^2 \leqslant \|f(uv)^k e_k'\|^2 \leqslant \varepsilon''$.

Lemma 6.1 shows that if n' is large enough and ε'' is small enough with $\varepsilon'' < (\varepsilon'/(n+1))^2$, then there exists a unitary element $w' \in N$ such that $\varphi(w'^k e) = 0$ for all $k \neq 0$ and $\|w' - uv\| < \varepsilon'/n + 1$. But then $\|fw'^k e_n\| \leqslant \Sigma_{p=0}^{k-1} \|f(uv)^p (w' - uv)(w)^{k-p-1} e_n\| + \|f(uv)^k e_n\| \leqslant k\varepsilon'/(n+1) + \varepsilon'/(n+1) = (k+1)\varepsilon'/(n+1) \leqslant \varepsilon'$, which proves $(*)$. Q.E.D.

7. End of the proof of Theorem 1.1: The type II_1 case. In this section we prove 1.1 in the case where N is of type II_1. By 2.11 and §5 this will end the proof of the theorem. We begin the section by reducing the problem in several steps to the case when the type II_1 von Neumann algebra N is separable, M is countable decomposable, and $N' \cap M$ contains no finite projections of M. Note from the beginning that by section 3 we may assume δ vanishes on a set of projections $\{p_i\}_i$ in the center of M having the properties $\Sigma p_i = 1$ and N_{p_i} is of countable type for each i.

7.1. *First reduction.* It is sufficient to prove the theorem for separable N (i.e., N with separable predual).

To show this, let $R \subset N$ be a copy of the hyperfinite type II_1 factor with the same unit as N (cf. [5]). There exists an increasing net of separable von Neumann subalgebras $\{N_i\}_i$ of N with $R \subset N_i$ and $\bar{\bigcup}_i N_i^w = N$. Indeed, if $\{p_j\}_{j \in J}$ is a partition of the unity in the center of N such that Np_j is countable decomposable for each j, then any countably generated von Neumann subalgebra of Np_j is separable, so that if N_i are such that $N_i p_j$ is countably generated and contains Rp_j for a finite number J_0 of $j \in J$ and if $N_i \Sigma_{j \notin J_0} p_j = R \Sigma_{j \notin J_0} p_j$, then N_i will do. Since $R \subset N_i$, each N_i is of type II_1, and if $K_i \in \mathscr{J}(M)$ is such that $\delta|_{N_i} = \operatorname{ad} K_i$, then by 4.6 there exists $T_i \in K_\delta$ (in fact, in $\overline{\mathrm{co}}^{\mathrm{w}}\{\delta(u)u^* | u$ unitary element of $N_i\} \subset \mathrm{K}_\delta$) such that $\operatorname{ad} T_i = \operatorname{ad} K_i = \delta|_{N_i}$. Let T be a weak limit point of $\{T_i\}_i$. Then $\operatorname{ad} T = \delta$ on $\bigcup N_i$, so that by the weak continuity of ad T and δ, $\operatorname{ad} T = \delta$ on $N = \bar{\bigcup} N_i^w$. Since N is of type II_1, by 4.4 we have $T \in \mathscr{J}(M)$.

7.2. *Second reduction.* It is sufficient to prove the theorem when N is separable and M is countable decomposable.

Indeed, by the preceding reduction we may assume N is separable. Let $\mathscr{U}_0$ be a countable subset in the unitary group $\mathscr{U}$ of N, dense in $\mathscr{U}$ in the $*$-strong operator topology. Let $\{p_i\}_{i\in J}$ be an increasing net of countable decomposable projections of M with $p_i \uparrow 1$. By the density of $\mathscr{U}_0$ in $\mathscr{U}$, it follows that for each i, $\vee\{up_iu^*|u \in \mathscr{U}\} = \vee\{up_iu^*|u \in \mathscr{U}_0\}$, so that if we denote this projection by s_i, then it is countable decomposable (being a supremum of a countable set of countable decomposable projections) and, moreover, $s_i \in N' \cap M$, $s_i \uparrow 1$. Define δ_i: $N_{s_i} \mapsto s_i\mathscr{J}(M)s_i = \mathscr{J}(M_{s_i})$ by $\delta_i(xs_i) = s_i\delta(x)s_i$. Since $s_i \in N' \cap M$, δ_i are well-defined derivations. If for each i there exists an element $K_i \in \mathscr{J}(M_{s_i})$ such that $\delta_i = \operatorname{ad} K_i$, then by 4.6 there exists $T_i \in K_\delta$ such that $s_iT_is_i \in K_{\delta_i} \subset s_iK_\delta s_i$ satisfies $\delta_i = \operatorname{ad}(s_iT_is_i)$. Let T be a weak limit point in M of the net $\{T_i\}_i$ ($\subset M$). Since $\{s_i\}_i$ converges strongly to the identity, $T \in K_\delta$ and $\operatorname{ad} T = \delta$ on N. Then 4.4 applies to get $T \in \mathscr{J}(M)$.

7.3. *Third reduction.* It is sufficient to prove the theorem when N is separable, M is countable decomposable, and $N' \cap M$ contains no finite projections of M.

Let $p_0 = \vee\{e' \in N' \cap M|e' \text{ finite projection of } M\}$. Note that in fact $p_0 = \vee\{e' \in N' \cap M|e' \text{ projection with } \varphi(e') < \infty\}$. Indeed, this follows immediately by 2.1, because given any $e \in N' \cap M$ and $p \in \mathscr{Z}(M)$ we have $ep \in N' \cap M$. Assume now that $\delta(x) = \delta(x)p_0$, $x \in N$. Then $\mathrm{K}_\delta = \mathrm{K}_\delta p_0$. For each unitary element $u \in N$ define on K_δ the weakly continuous affine transformation $\mathrm{T}_u(x) = uxu^* + \delta(u)u^*$. Then $\mathrm{T}_u\mathrm{T}_v = \mathrm{T}_{uv}$, and since $\mathrm{T}_u(\delta(v)v^*) = u\delta(v)v^*u^* + \delta(u)u^* = \delta(uv)v^*u^*$, it follows that $\mathrm{T}_u(\mathrm{K}_\delta) \subset \mathrm{K}_\delta$. Consider on M the seminorms $\mathscr{S} = \{\varphi(x^*xe')^{1/2} \text{ for } x \in M|e' \text{ finite projection in } N' \cap M \text{ with } \varphi(e') < \infty\}$. Then the semigroup of transformations T_u on K_δ is noncontractive, because if $x, y \in \mathrm{K}_\delta$, $x \neq y$, then $\inf_u\varphi(u(x-y)^*(x-y)u^*e') = \varphi((x-y)^*(x-y)e')$, and if $\varphi((x-y)^*(x-y)e') = 0$, then $x - y = (x-y)p_0 = (x-y)(\vee e') = 0$ (by the faithfulness of φ). Thus by the Ryll-Nardjewski fixed point theorem (see A.3 in [9]) there exists an element $X \in K_\delta$ with $\mathrm{T}_u(X) = X$ for all unitary elements $u \in N$. But then $uXu^* + \delta(u)u^* = X$ and thus $\delta(u) = Xu - uX$, and by linearity $\delta(x) = Xx - xX$ for all $x \in N$. Since N is of type II_1, by 4.4 we get $X \in \mathscr{J}(M)$. Similarly, if $\delta(x) = p_0\delta(x)$, for any $x \in N$ we obtain that δ is implemented by an element in $\mathscr{J}(M)$. It follows that there exists $K \in \mathscr{J}(M)$ such that $(\delta - \operatorname{ad} K)(x) = (1-p_0)(\delta - \operatorname{ad} K)(x)(1-p_0)$. Thus, if we define δ_0: $N_{1-p_0} \mapsto M_{1-p_0}$ by $\delta_0(x(1-p_0)) = (\delta - \operatorname{ad} K)(x)(1-p_0)$, then δ_0 is a well-defined derivation taking values into $(1-p_0)\mathscr{J}(M)(1-p_0) = \mathscr{J}(M_{1-p_0})$. Since $N'_{1-p} \cap M_{1-p_0}$ contains no finite projections of M_{1-p_0} (see the proof of 6.2, (2)), this shows that in order to prove the theorem for N separable of type II_1 and M countable decomposable, we may in addition assume that $N' \cap M$ contains no finite projections of M.

7.4. In the rest of this section we may therefore assume N is separable, M is of countable type, and $N' \cap M$ contains no finite projections of M.

By 6.2 we may construct subalgebras $A \subset R \subset N$ so that R is an approximately finite-dimensional type II_1 von Neumann subalgebra of N, A is a maximal abelian von Neumann subalgebra in N, and $A' \cap M$ contains no finite projections of M. Since R is approximately finite-dimensional, there exists an amenable subgroup of unitary elements $\mathscr{U}$ in R such that $\mathscr{U}'' = R$. Let $K = \int_{\mathscr{U}} \delta(u)u^* \, d\mu(u)$, where μ is an invariant mean on $\mathscr{U}$. Then, like 2.12, it is easy to see that ad K equals δ on $\mathscr{U}$ and thus on R. By 4.4 it follows that $K \in \mathscr{J}(M)$. Thus, by taking $\delta - \operatorname{ad} K$ instead of δ if necessary, we may suppose δ vanishes on R and thus on $A \subset R$.

We show that $\delta = 0$ on all N follows from the fact that $\delta|_A = 0$, and this will end the proof of Theorem 1.1.

Assume $\delta \neq 0$. Then there exists a unitary element $v \in M$ such that $\delta(v) \neq 0$. Moreover, there exists a finite projection $e \in M^1_{\varphi,\psi}$ such that $\varphi(ev^*\delta(v)e) \neq 0$, because otherwise $\varphi(v^*\delta(v)x) = 0$ for any linear combination x of projections $e \in M^1_{\varphi,\psi}$ and thus, by taking norm limits, for any $x \in M_\varphi$, which implies $v^*\delta(v) = 0$, a contradiction.

Let $q \in N$ be the support of the normal form $N \ni y \mapsto \varphi(ye)$. Then $qe = e$ and thus there are central projections p_n in N so that

(*) p_n increases to the central support of g in N;

(**) for each n there is a finite number of unitary elements $u_1, \ldots, u_{k(n)}$ in N so that

$$\bigvee_i u_i p_n q u_i^* = p_n.$$

Now from (*) it follows that if n is large enough, then $\|e - p_n e p_n\|_\varphi$ is small enough to ensure that $\varphi(p_n e p_n v^* \delta(v) p_n e p_n) \neq 0$. Since A is maximal abelian in N, $A \supset \mathscr{Z}(N)$, so that δ vanishes on $\mathscr{Z}(N)$ and thus on all p_n. Moreover, if δ_n: $Np_n \to \mathscr{J}(p_n M p_n)$ is defined by $\delta_n(xp_n) = p_n \delta(x) p_n$, then δ_n vanishes on Ap_n and the support projection e_n of $\sum_i u_i p_n e p_n u_i^*$ (where u_i are as in (**)) satisfies $\varphi(e_n(vp_n)^*\delta_n(vp_n)e_n) \neq 0$; $Np_n \ni y \mapsto \varphi(ye_n)$ is faithful on Np_n; e_n is a finite sum of elements in $M^1_{\varphi,\psi}$.

Altogether, these considerations show that, by modifying N, M, δ, φ, and ψ, if necessary we may suppose we are in the following situation:

(i) M is a countable decomposable semifinite von Neumann algebra with a normal semifinite faithful trace φ constructed from a normal faithful state ψ on $\mathscr{Z}(M)$ as in 2.1.

(ii) $N \subset M$ is a separable type II_1 von Neumann subalgebra and $A \subset N$ is a maximal abelian von Neumann subalgebra of N such that $A' \cap M$ has no finite projections of M.

(iii) δ: $N \to \mathscr{J}(M)$ is a derivation that vanishes on A.

(iv) $v \in N$ is a unitary element and $e \in M$ is a finite projection satisfying the following properties:

(1) There exists a constant $c \geqslant 1$ such that $\psi(q) \leqslant \varphi(eq) \leqslant c\psi(q)$ for any $q \in \mathscr{Z}(M)$.

(2) $\varphi(ev^*\delta(v)e) = 1$ (by suitable amplification of δ with a scalar).

(3) $N \ni y \mapsto \varphi(ye)$ is faithful.

We now prove that for any n there exist a finite projection $e_n \in M$ and a unitary element $w_n \in N$ such that

(a) $e_n \leqslant e, \varphi(e - e_n) \leqslant 2^{-n}$;

(b) $\|e_n w_n^k e_n\| < 2^{-n}$ for $k \neq 0$, $|k| \leqslant n$;

(c) $\varphi(e_n w_n^k e_n) = 0$ for $k \neq 0$;

(d) $|\varphi(e_n w_n^{-p}\delta(w_n^p)e_n) - 1| < 2^{-n}$ if $n \geqslant p > 0$; and

(e) $|\varphi(e_n w_n^{-s}\delta(w_n^p)e_n)| < 2^{-n}$ if $p \neq s$ or $p \leqslant 0$, $|p|, |s| \leqslant n$.

To do this, let $f_0 \in M$ be a finite projection such that

$$\|\delta(v)(1 - f_0)\| < (5cn)^{-1}2^{-n-1}, \|(1 - f_0)v^{-1}\delta(v)\| < (5cn)^{-1}2^{-n-1}, \|\delta(v^{-1})v(1 - f_0)\| < (5cn)^{-1}2^{-n-1}.$$

By 2.1 there exists a central projection $p \in \mathscr{Z}(M)$ such that $\varphi(f_0 p) < \infty$ and $\psi(1 - p) \leqslant (5c(\|\delta\| + 1)2^{n+1})^{-2}$ (in all these inequalities $c \geqslant 1$ is the constant appearing in (*iv*)). Set $f = f_0 p \vee e$. Then by the preceding Corollary 6.4 there exist unitary elements $w_n \in N$ and $u_n \in A$ and a projection $e_n \in M$ such that

(a′) $e_n \leqslant e, \quad \varphi(e - e_n) \leqslant 2^{-n}$;

(b′) $\|f w_n^k e_n\| \leqslant (5cn(\|\delta\| + 1) + 1)^{-1}2^{-n-1}$ for $|k| \leqslant 2n$, $k \neq 0$;

(c′) $\|w_n - u_n v\| \leqslant (5cn(\|\delta\| + 1))^{-1}2^{-n-1}$ and $\varphi(e_n w_n^k e_n) = 0$ for $k \neq 0$.

It follows that if $n \geqslant k > 0$, then

(i) $$\|\delta(w_n^k)e_n - w_n^{k-1}\delta(w_n)e_n\|_\varphi$$

$$= \left\| \sum_{s=0}^{k-2} w_n^s \delta(w_n) w_n^{k-s-1} e_n \right\|_\varphi \leqslant \sum_{s=0}^{k-2} \|\delta(w_n) w_n^{k-s-1} e_n\|_\varphi$$

$$\leqslant \sum_{s=0}^{k-2} \|\delta(u_n v) w_n^{k-s-1} e_n\|_\varphi + (k-1)\|\delta\| \, \|e\|_\varphi \|w_n - u_n v\|$$

$$= \sum_{s=0}^{k-2} \|\delta(v) w_n^{k-s-1} e_n\|_\varphi + (k-1)c^{1/2}\|\delta\| \, \|w_n - u_n v\|$$

$$\leqslant \sum_{s=0}^{k-2} \|\delta(v) f w_n^{k-s-1} e_n\|_\varphi + (k-1)\|\delta\| \|(1-p)e_n\|_\varphi$$

$$+(k-1)c^{1/2}\|\delta(v)(1-f_0)\| + 5^{-1}c^{-1/2}2^{-n-1} \leqslant 2^{-n-1}c^{-1/2};$$

(ii) $$\|\delta(w_n^{-k})e_n\|_\varphi \leqslant \sum_{s=0}^{k-1} \left\|(w_n^{-1})^s \delta(w_n^{-1}) (w_n^{-1})^{k-s-1} e_n\right\|_\varphi$$

$$\leqslant \sum_{s=0}^{k-1} \left\|(\delta(v^{-1})v)(u_n v)^{-1}(w_n^{-1})^{k-s-1} e_n\right\|_\varphi$$

$$+c^{1/2}k\|\delta\| \, \|u_n v - w_n\|$$

$$\leqslant \sum_{s=0}^{k-1} \left\|\delta(v^{-1})v(w_n^{-1})^{k-s} e_n\right\|_\varphi + 2kc^{1/2}\|\delta\| \, \|u_n v - w_n\|$$

$$\leqslant \sum_{s=0}^{k-1} \left\|\delta(v^{-1})vf(w_n^{-1})^{k-s} e_n\right\|_\varphi + c^{-1/2}k(5n)^{-1}2^{-n-1}$$

$$+k\|\delta\|\|(1-p)e_n\|_\varphi + (2/5)c^{-1/2}2^{-n-1}$$

$$\leqslant \|\delta\| \sum_{s=0}^{k-1} \left\|f(w_n^{-1})^{k-s} e_n\right\|_\varphi + (4/5)2^{-n-1}c^{-1/2} \leqslant 2^{-n-1}c^{-1/2}.$$

Thus for $n \geqslant p > 0$ we have by (i), (c′), and the equality $\delta(u_n v) = u_n \delta(v)$:

$$\begin{aligned}|\varphi(e_n w_n^{-p}\delta(w_n^p)e_n) - 1| &\leqslant |\varphi(e_n w_n^{-1}\delta(w_n)e_n) - 1| + 2^{-n-1}\\ &\leqslant |\varphi(e_n v^{-1}u_n^{-1}\delta(u_n v)e_n) - 1| + 2c^{1/2}\|\delta\|\,\|w_n - u_n v\|\\ &\quad + 2^{-n-1}\\ &\leqslant |\varphi(ev^{-1}\delta(v)e) - 1| + 2^{-n} = 2^{-n}.\end{aligned}$$

If $n \geqslant p > 0$ and $s \neq p$, then by (i), (c′), and (b′) we have:

$$\begin{aligned}|\varphi(e_n w_n^{-s}\delta(w_n^p)e_n)| &\leqslant |\varphi(e_n w_n^{-s+p-1}\delta(w_n)e_n)| + 2^{-n-1}\\ &\leqslant |\varphi(e_n w_n^{-s+p-1}u_n v v^{-1}\delta(v)e_n)| + 5^{-1}2^{-n-1} + 2^{-n-1}\\ &\leqslant |\varphi(e_n w_n^{-s+p}v^{-1}\delta(v)e_n)| + 2.5^{-1}2^{-n-1} + 2^{-n-1}\\ &\leqslant |\varphi(e_n w_n^{-s+p}fv^{-1}\delta(v)e_n)| + 4.5^{-1}.2^{-n-1} + 2^{-n-1}\\ &\leqslant (5^{-1} + 4.5^{-1} + 1)2^{-n-1} = 2^{-n}.\end{aligned}$$

Finally, if $p < 0$, then by (ii) and the Cauchy-Schwartz inequality we have for any s:

$$|\varphi(e_n w_n^{-s}\delta(w_n^p)e_n)| \leqslant \|\delta(w_n^p)e_n\|_\varphi \|e_n\|_\varphi \leqslant 2^{-n}.$$

This shows that e_n and w_n as defined before fulfill conditions (a)–(e).

We now define $A_n \subset N$ to be the von Neumann algebra generated by w_n; $p_n \in \mathscr{B}(L^2(M, \varphi))$ to be the orthogonal projections onto $\overline{A_n e_n}$; the isometries u_n: $L^2(\mathbb{T}, \mu) \mapsto L^2(M, \varphi)$ (where μ is the normalized Lebesgue measure on the torus $\mathbb{T}$) to be defined by $u_n(z^k) = \varphi(e_n)^{-1/2} w_n^k e_n$ and the measure preserving isomorphism Ψ_n: $L^\infty(T, \mu) \mapsto (A_n, \varphi(e_n)^{-1}\varphi(.e_n))$ by $\Psi_n(z^k) = w_n^k$. Moreover, we define δ_n: $L^\infty(\mathbb{T}, \mu) \mapsto \mathscr{B}(L^2(\mathbb{T}, \mu))$ by $\delta_n(f) = u_n^*\delta(\Psi_n(f))u_n$ for $f \in L^\infty(\mathbb{T}, \mu)$. Since $p_n = u_n u_n^* \in A_n'$, an easy computation shows that all δ_n are derivations and clearly $\|\delta_n\| \leqslant \|\delta\|$.

Let ω be a free ultrafilter on $\mathbb{N}$ and denote Δ: $L^\infty(\mathbb{T}, \mu) \mapsto \mathscr{B}(L^2(\mathbb{T}, \mu))$ by $\Delta(f) = w - \lim_{n \mapsto \omega}\delta_n(f)$. Then Δ is also a derivation and $\|\Delta\| \leqslant \|\delta\|$. We show that if p denotes the orthogonal projection onto the Hardy space $H^2(\mathbb{T}, \mu) = \overline{\operatorname{span}}\{z^k | k > 0\} \subset L^2(\mathbb{T}, \mu)$, then $\Delta = \operatorname{ad} P$ and Δ is a continuous function from the unit ball of $L^\infty(\mathbb{T}, \mu)$ with the norm $\| \ \|_2$ into $\mathscr{B}(L^2(\mathbb{T}, \mu))$ with the uniform norm. To prove the first assertion, note that by (4) $\langle \delta_n(z^p)1, z^s \rangle = \varphi(e_n w_n^{-s}\delta(w_n^p)e_n)$ tend to 1 for $p = s > 0$ and to 0 otherwise, so that $\langle \Delta(z^p)1, z^s \rangle$ is equal to 1 if $p = s > 0$ and to 0 otherwise. Since $\operatorname{ad} P$ also satisfies these equalities and Δ, $\operatorname{ad} P$ are derivations, it follows that $\langle \Delta(z^p)z^k, z^s \rangle =$

$\langle \text{ad } P(z^p)z^k, z^s\rangle$ for all $k, p, s \in \mathbb{Z}$, and thus, by linearity and weak continuity of Δ and ad P, $\Delta = \text{ad } P$.

To prove the second assertion (i.e., the continuity result for Δ), note first that

(∗) given $\beta > 0$, there exists $n_0 \geqslant 1$ and $\alpha > 0$ such that for any $n \geqslant n_0$ and $a \in A_n$, with $\|a\| \leqslant 1$ and $\varphi(e_n a^* a e_n) < \alpha$ we have $|||\delta(a)||| < \beta$.

Indeed, since $N \ni x \mapsto \varphi(xe)$ is faithful on N by 4.1, there exists $\alpha' > 0$ such that if $a \in N$, $\|a\| \leqslant 1$, $\varphi(ea^*ae) < \alpha'$, then $|||\delta(a)||| < \beta$. Let n_0 be such that if $n \geqslant n_0$, then $\varphi(e - e_n) < \alpha'/2$. If we take $\alpha = \alpha'/2$ and if $\varphi(e_n a^* a e_n) \leqslant \alpha$, then we get $\varphi(ea^*ae) \leqslant \varphi(e - e_n)\|a^*a\| + \alpha \leqslant \alpha'/2 + \alpha'/2 = \alpha'$, so that $|||\delta(a)||| < \beta$.

Now the required continuity assertion on Δ states that given any $\beta > 0$ there exists $\alpha > 0$ such that if $f \in L^\infty(\mathbb{T}, \mu)$, $\|f\| \leqslant 1$, and $\|f\|_2 < \alpha$, then $\|\Delta(f)\xi\|_2 < \beta$ for any $\xi \in L^2(\mathbb{T}, \mu)$, $\|\xi\|_2 \leqslant 1$. In fact it is sufficient to check this for ξ Laurent polynomials, $\xi = \sum_{|k| \leqslant m} \alpha_k z^k$ (with $\sum |\alpha_k|^2 \leqslant 1$). Let α be the one given by (∗). Then if $a_n = \Psi_n(f)$, we have

$$\begin{aligned}
\|\Delta(f)\xi\|_2 &\leqslant \limsup_n \|\delta_n(f)\xi\|_2 \\
&= \limsup_n \left\| p_n \delta(a_n) p_n \Big(\sum_{|k| \leqslant m} \alpha_k w_n^k \Big) e_n \right\|_\varphi \varphi(e_n)^{-1/2} \\
&\leqslant \limsup_n \left\| \delta(a_n) \Big(\sum_{|k| \leqslant m} \alpha_k w_n^k \Big) e_n \right\|_\varphi \varphi(e_n)^{-1/2}.
\end{aligned}$$

But $\|(\sum_{|k| \leqslant m} \alpha_k w_n^k) e_n\|^2 = \|\sum_{i,j} \bar{\alpha}_i \alpha_j e_n w_n^{j-i} e_n\| \leqslant \sum_i |\alpha_i|^2 + \sum_{i \neq j} |\alpha_i|\,|\alpha_j|\,\|e_n w_n^{j-i} e_n\|$, and since $\sum_{i,j} |\alpha_i|\,\|\alpha_j\| = (\sum |\alpha_i|)^2 \leqslant (2m+1)\sum |\alpha_i|^2 \leqslant 2m+1$, by (b) we get $\|(\sum \alpha_k w_n^k) e_n\|^2 \leqslant 1 + (2m+1)2^{-n}$. Thus, since for $n \geqslant n_0$ we have $|||\delta(a_n)||| < \beta$, it follows that if $n \geqslant n_0$, $\|\delta(a_n)(\sum_{|k| \leqslant m} \alpha_k w_n^k) e_n\|_\varphi \leqslant (1 + (2m+1)2^{-n})^{1/2} \beta \varphi(e_n)^{1/2}$. Hence $\lim_n \sup \|\delta(a_n)(\sum \alpha_k w_n^k) e_n\|_\varphi \leqslant \beta \varphi(e_n)^{1/2}$ and thus $\|\Delta(f)\xi\|_2 \leqslant \beta$.

We have thus proved that ad P is continuous from the unit ball of $L^\infty(\mathbb{T}, \mu)$ with the two-norm into $\mathscr{B}(L^2(\mathbb{T}, \mu))$ with the uniform norm. But ad P takes values into the finite rank operators for all the polynomials in $L^2(\mathbb{T}, \mu)$, so that by the above continuity it follows that ad P takes values into $\mathscr{K}(L^2(\mathbb{T}, \mu))$ on all $L^\infty(\mathbb{T}, \mu)$. But then by §5 (the abelian case of the theorem) ad P is equal to ad K for some $K \in \mathscr{K}(L^2(\mathbb{T}, \mu))$. It follows that $P - K \in L^\infty(\mathbb{T}, \mu)$ and thus $P - K$ is a multiplication operator M_f for some function $f \in L^\infty(\mathbb{T}, \mu)$ (since $L^\infty(\mathbb{T}, \mu)$ is maximal abelian in $\mathscr{B}(L^2(\mathbb{T}, \mu))$. But $1 = \lim_{n \to \infty} \langle (P - K) z^n, z^n \rangle = \int z^{-n} f z^n \, d\mu(z) = \int f \, d\mu(z) = \lim_{n \mapsto -\infty} \langle (P - K) z^n, z^n \rangle = 0$, which is a contradiction.

The initial assumption $\delta \neq 0$ is therefore false and so Theorem 1.1 is completely proved.

8. The counterexample: Proof of Theorem 1.2. The most simple yet typical situation when the condition of local compatibility between N and $\mathscr{Z}(M)$ is not fulfilled, for abelian (or, more generally, finite type I) N algebras, is when M is the algebra $L^\infty([0,1], \lambda) \overline{\otimes} \mathscr{B}(L^2(\mathbb{T}, \mu))$ and $N = 1 \otimes L^\infty(\mathbb{T}, \mu)$, where μ is the Lebesgue measure on the torus $\mathbb{T}$ and λ is the Lebesgue measure on the unit interval $[0,1]$.

It is well known that $L^\infty([0,1], \lambda) \overline{\otimes} \mathscr{B}(L^2(\mathbb{T}, \mu))$ can be identified with $L^\infty([0,1], \mathscr{B}(L^2(\mathbb{T}, \mu))$ with L^∞ having here the obvious significance (i.e., weak λ-measurable functions of $[0,1]$ into $\mathscr{B}(L^2(\mathbb{T}, \mu))$, uniformly bounded, considered modulo a.e. vanishing such functions). Under this identification the ideal $\mathscr{J}(M)$ may be identified with the functions in $L^\infty([0,1], \mathscr{B}(L^2(\mathbb{T}, \mu))$ which take values a.e. in $\mathscr{K}(L^2(\mathbb{T}))$. We denote this set by $L^\infty([0,1], \mathscr{K}(L^2(\mathbb{T}))$. The subalgebra $N = 1 \otimes L^\infty(\mathbb{T}, \mu)$ in turn becomes the algebra of all constant, $L^\infty(\mathbb{T}, \mu)$-valued functions on $[0,1]$. Moreover, the center of M may be identified with the scalar-valued functions on $[0,1]$, i.e., $\mathscr{Z}(M) = L^\infty([0,1], \mathbb{C}1_{L^2(\mathbb{T})})$.

Note also that the von Neumann algebra generated by N and $\mathscr{Z}(M)$ is $\tilde{N} = L^\infty([0,1], L^\infty(\mathbb{T}, \mu)) \subset M = L^\infty([0,1], \mathscr{B}(L^2(\mathbb{T}, \mu)))$ (in tensor product terms it equals $L^\infty([0,1] \overline{\otimes} L^\infty(\mathbb{T}))$.

Now a general observation concerning problems on derivations into $\mathscr{J}(M)$: By Theorem 1.1, if the von Neumann subalgebra N contains the center of M, then any derivation δ of N into $\mathscr{J}(M)$ is implemented by an element in $\mathscr{J}(M)$; thus, if N does not contain $\mathscr{Z}(M)$, it is natural to try to show that the unique extension of δ to the von Neumann algebra generated by N and $\mathscr{Z}(M)$ still take values into $\mathscr{J}(M)$. It turns out that this is not always the case. More precisely, we will construct an element $T \in M = L^\infty([0,1], \mathscr{B}(L^2(\mathbb{T}, \mu)))$ so that $[T, N] \subset \mathscr{J}(M)$ but so that $[T, \tilde{N}] \not\subset \mathscr{J}(M)$. Then if $K \in \mathscr{J}(M)$ were such that $T - K \in N' \cap M$, it would follow that $\operatorname{ad} T = \operatorname{ad} K$ on $\tilde{N}$, so that $[T, \tilde{N}] \subset \mathscr{J}(M)$, a contradiction.

The key point of the construction of an element T as above is the following:

8.1. Lemma. *There exists $T_0 \in \mathscr{B}(L^2(\mathbb{T}, \mu))$ such that:*

(1) *Given any measurable set $E \subset \mathbb{T}$ with $1 \in \mathbb{T}$ a point of Lebesgue density 0 or 1 for E, the projection $e = \chi_E \in L^\infty(\mathbb{T}, \mu)$ satisfies $[T_0, e] \in \mathscr{K}(L^2(\mathbb{T}, \mu))$.*

(2) *There exists a projection $e_0 \in L^\infty(\mathbb{T}, \mu)$ such that $[T_0, e_0] \notin \mathscr{K}(L^2(\mathbb{T}, \mu))$.*

Before proving this lemma, let us show how one can construct the desired element T in M from the operator T_0.

8.2. Proposition. *Let U be the unitary element in $M = L^\infty([0,1], \mathscr{B}(L^2(\mathbb{T}, \mu)))$ defined by $U = (U_t)_{0 \leq t \leq 1}$ with $U_t\colon L^2(\mathbb{T}, \mu) \mapsto L^2(\mathbb{T}, \mu)$, $(U_t f)(e^{2\pi i x}) = f(e^{2\pi i (x+t)})$, $x \in [0,1]$. Let $\tilde{T}_0$ be the element T_0 of Lemma 8.1 regarded as a constant function in M (i.e., $\tilde{T}_0 = 1 \otimes T_0$). Then $T = U\tilde{T}_0U^*$ satisfies $[T, N] \subset \mathscr{J}(M)$, but there exists no $K \in \mathscr{J}(M)$ so that $\operatorname{ad} T = \operatorname{ad} K$ on N.*

Proof of 8.2. To prove that $[T, N] \subset \mathscr{J}(M)$ it is sufficient to show that $[T, \tilde{e}] \in \mathscr{J}(M))$ for any projection $\tilde{e} \in N = 1 \otimes L^\infty(\mathbb{T}, \mu)$. Thus we have to show that given any projection $e \in L^\infty(\mathbb{T}, \mu)$ we have $[U_t T_0 U_t^*, e] \in \mathscr{K}(L^2(\mathbb{T}, \mu))$ for λ-almost all $t \in [0,1]$.

Now if $e = \chi_E$ for some measurable subset $E \subset \mathbb{T}$, then by Lebesgue's theorem for almost all $t \in [0,1]$, $e^{2\pi i t}$ has density 0 or 1. But if t is so that $e^{2\pi i t} \in \mathbb{T}$ is a point of density 0 or 1 in E, then the set E_t corresponding to the projection $U_t^* e U_t$ (i.e., $\chi_{E_t} = U_t^* e U_t$) has density 0 or 1 in the point $1 \in \mathbb{T}$. Thus by 8.1, (1), $[T_0, U_t^* e U_t] \in \mathscr{K}(L^2(\mathbb{T}, \mu))$, which shows that $[U_t T_0 U_t^*, e] \in \mathscr{K}(L^2(\mathbb{T}, \mu))$.

This shows that $[U_t T_0 U_t^*, e] \in \mathscr{K}$ λ-a.e. in $t \in [0,1]$ and proves that $T = U\tilde{T}_0 U^*$ satisfies $[T, N] \subset \mathscr{K}(L^2(\mathbb{T}, \mu))$.

Now if $K \in \mathscr{J}(M) = L^\infty([0,1], \mathscr{K}(L^2(\mathbb{T}, \mu)))$ is such that ad T = ad K on N, then, since the elements in $\mathscr{Z}(M)$ commute with both T and K, it follows that ad T = ad K on the von Neumann algebra $\tilde{N} = L^\infty([0,1], L^\infty(\mathbb{T}, \mu))$ generated by N and $\mathscr{Z}(M)$. But $U\tilde{N}U^* = \tilde{N}$ and more precisely $\tilde{e} = (U_t e_0 U_t^*)_{0 \leqslant t \leqslant 1}$ is in $L^\infty([0,1], L^\infty(\mathbb{T}, \mu))$, so that $[T, \tilde{e}] = [K, \tilde{e}] \in \mathscr{J}(M)$, which means that $[T_0, e_0] = [U_t T_0 U_t^*, U_t e_0 U_t^*] \in \mathscr{K}(L^2(\mathbb{T}, \mu))$ for λ-almost all $t \in [0,1]$. But this contradicts 8.1, (2). Q.E.D.

Proof of 8.1. Let A_n, B_n be subsets of T defined as follows: $A_n = \exp(2\pi i[1/2^{2n}, 1/2^{2n-1}])$, $B_n = \exp(2\pi i[1/2^{2n+1}, 1/2^{2n}])$, for each $n \geqslant 1$.

For an element $f \in L^2(\mathbb{T}, \mu)$ we denote by $\|f\|_2$ its norm. We define $\xi_n = \|\chi_{A_n}\|_2^{-1}\chi_{A_n} \in L^2(\mathbb{T}, \mu)$ and $\eta_n = \|\chi_{B_n}\|_2^{-1}\chi_{B_n} \in L^2(\mathbb{T}, \mu)$.

Note that $\{\xi_n\}_n \cup \{\eta_n\}_n$ is an orthonormal family of vectors in $L^2(\mathbb{T}, \mu)$.

If $\xi, \eta \in L^2(\mathbb{T}, \mu)$, we denote by $p_{\xi,\eta}$ the one-dimensional operator in $\mathscr{B}(L^2(\mathbb{T}, \mu))$ defined by

$$p_{\xi,\eta}(\zeta) = \langle \zeta, \eta \rangle \xi.$$

We define $T_0 = \Sigma_n p_{\xi_n, \eta_n}$ (the infinite sum is so-convergent because ξ_n, η_m are all mutually orthogonal vectors). Note that in fact T_0 is a partial isometry with $T_0^2 = 0$.

Let $E \subset \mathbb{T}$ be a measurable set of density zero in 1. We show that $e = \chi_E$ satisfies $eT_0, T_0 e \in \mathscr{K}(L^2(\mathbb{T}, \mu))$. Indeed, we have $eT_0 = \Sigma p_{\chi_E \xi_n, \eta_n}$. Since the vectors $\{\chi_E \xi_n\}_n$ are mutually orthogonal in $L^2(\mathbb{T}, \mu)$, to show that eT_0 is compact it is sufficient to show that $\|\chi_E \xi_n\|_2 \mapsto 0$. But

$$\|\chi_E \xi_n\|_2^2 = \frac{\mu(E \cap A_n)}{\mu(A_n)} \leqslant \frac{\mu\big(E \cap \exp\big(2\pi i\big[-1/2^{2n-1}, 1/2^{2n-1}\big]}{\mu(A_n)}$$

$$= 4\frac{\mu\big(E \cap \exp\big(2\pi i\big[-1/2^{2n-1}, 1/2^{2n-1}\big]\big)}{\mu\big(\exp\big(2\pi i\big[-1/2^{2n-1}, 1/2^{2n-1}\big]\big)\big)},$$

and this last term tends to zero, because 1 is point of density zero for E. Similarly, we have $T_0 e = \Sigma p_{\xi_n, \chi_E \eta_n} \in \mathscr{K}(L^2(\mathbb{T}, \mu))$.

Moreover, if e corresponds to a set of density one in 1, then by the above $[T_0, 1 - e] \in \mathscr{K}(L^2(\mathbb{T}, \mu))$, so that $[T_0, e]$ is also compact.

Now to show that T_0 also satisfies condition 8.1, (2), let $E_0 \subset \mathbb{T}$ be a measurable set so that $\mu(E_0 \cap A_n) = \mu(A_n)/2$ and $\mu(E_0 \cap B_n) = \mu(B_n)/2$ (e.g., take E_0 to be the union of the halves of each interval A_n or B_n).

It is easy to see that if $e_0 = \chi_{E_0}$, then $[T_0, e_0] = \sum_n p_{(1-\chi_{E_0})\xi_n, \chi_{E_0}\eta_n} - \sum_n p_{\chi_{E_0}\xi_n, (1-\chi_{E_0})\eta_n}$. Moreover, the vectors

$$\{\chi_{E_0}\xi_n, (1 - \chi_{E_0})\xi_n, \chi_{E_0}\eta_n, (1 - \chi_{E_0})\eta_n | n \geqslant 1\}$$

are all mutually orthogonal. Thus, to prove that $[T_0, e_0] \notin \mathscr{K}(L^2(\mathbb{T}, \mu))$ it is sufficient to show that $\{\|\chi_{E_0}\xi_n\|_2^2 \|(1 - \chi_{E_0})\eta_n\|_2^2\}_n$ does not tend to zero. But

$$\|\chi_{E_0}\xi_n\|_2^2 \|(1 - \chi_{E_0})\eta_n\|_2^2 = \frac{\mu(E_0 \cap A_n)}{\mu(A_n)} \frac{\mu(B_n \cap E_0)}{\mu(B_n)} = 1/4.$$

Thus $[T_0, e_0] \notin \mathscr{K}(\mathrm{L}^2(\mathbb{T}, \mu))$, which ends the proof of 8.1. Q.E.D.

8.3. *Final remarks.* Theorem 1.2 suggests that in all the cases left uncovered by Theorem 1.1 the derivation problem into the compacts has a negative answer. In fact, with some extra effort one may easily extend the methods of this section to get counterexamples in a large class of cases. However, let us point out here one case left open which deserves attention and for which we could not construct a counterexample:

8.3.1. *Problem.* Let M_0 be a type II_1 factor, $\mathscr{H}$ an infinite-dimensional Hilbert space and $M = \mathrm{L}^\infty([0,1], \lambda) \overline{\otimes} M_0 \overline{\otimes} \mathscr{B}(\mathscr{H})$. Let $A_0 \subset M_0$ be a (maximal) abelian $*$-subalgebra of M_0 and $A_1 \subset \mathscr{B}(\mathscr{H})$ an atomic (maximal) abelian $*$-subalgebra of $\mathscr{B}(\mathscr{H})$. Let $N_0 = 1 \otimes A_0 \otimes 1$ and $N = 1 \otimes A_0 \otimes A_1$. Is it true that any derivation of N_0 (or of N) into $\mathscr{J}(M)$ is inner?

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