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From non-linear elasticity to linear elasticity with initial stress via Γ-convergence

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Abstract We consider a initially stressed hyperelastic body in equilibrium in its undeformed configuration under a system of dead loads. We give sufficient conditions on the stored energy which guarantee that when the loads undergo a small perturbation, the energy functional Γ converges, after some re-scaling, to the energy functional of linear elasticity with initial stress. We also show, under stronger conditions, that quasi-minimizers of the non-linear problem converge to a minimizer of the incremental problem.

Keywords Initial stress · Incremental elasticity

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1 Introduction

Consider a continuous body occupying, in its reference configuration, a domain Ω of $\mathbb{R}^3$ with boundary $\partial\Omega$ having outward unit normal $\mathbf{n}$. Let $\mathring{\mathbf{T}}$ be the stress field in the reference configuration, the so-called *initial stress*. We make no assumptions on what might have given rise to the initial stress, we just mention that it might be the result of prior inelastic deformation. We suppose that the body be clamped on a part $\partial_D\Omega$ of the boundary $\partial\Omega$, and that the body be in equilibrium under the external loads $\mathring{\mathbf{d}}\colon \Omega \to \mathbb{R}^3$ and $\mathring{\mathbf{c}}\colon \partial_N\Omega \to \mathbb{R}^3$, with $\partial_N\Omega = \partial\Omega\backslash\partial_D\Omega$, satisfying:

$$\int_\Omega \mathring{\mathbf{T}} \cdot \nabla\boldsymbol{\varphi}\, \mathrm{d}\mathcal{L}^3 = \int_\Omega \mathring{\mathbf{d}} \cdot \boldsymbol{\varphi}\, \mathrm{d}\mathcal{L}^3 + \int_{\partial_N\Omega} \mathring{\mathbf{c}} \cdot \boldsymbol{\varphi}\, \mathrm{d}\mathcal{H}^2 \tag{1}$$

for every smooth displacement field $\boldsymbol{\varphi} : \Omega \to \mathbb{R}^3$ taking null values on $\partial_D\Omega$. If (1) holds with $\mathring{\mathbf{d}}= \mathbf{0}$, $\mathring{\mathbf{c}}= \mathbf{0}$, and $\mathring{\mathbf{T}}\,\mathbf{n} = \mathbf{0}$ on $\partial_D\Omega$, then the initial stress is said to be *residual*. Suppose, in addition, that the body be hyperelastic. Then,

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$$\overset{\circ}{\mathbf{T}}(x) = \partial \mathsf{w}(x, \mathbf{0}), \tag{2}$$

where $\mathsf{w}(x, \mathbf{E})$ is the stored energy density at the typical point x in Ω, expressed in terms of the Green–Saint Venant strain tensor: $\mathbf{E}(\mathbf{u}) = \mathrm{sym}\nabla\mathbf{u} + \frac{1}{2}\nabla\mathbf{u}^{\mathsf{T}}\nabla\mathbf{u}$. Let the reference be an equilibrium configuration. Then, at least formally, the null displacement $\mathbf{u} \equiv \mathbf{0}$ is a stationary point of the energy

$$\mathsf{E}(0; \mathbf{u}) = \int_{\Omega} \mathsf{w}(\cdot, \mathbf{E}(\mathbf{u}))\, \mathrm{d}\mathcal{L}^3 - \overset{\circ}{\mathsf{L}}[\mathbf{u}],$$

where

$$\overset{\circ}{\mathsf{L}}[\mathbf{u}] = \int_{\Omega} \overset{\circ}{\mathbf{d}} \cdot \mathbf{u}\, \mathrm{d}\mathcal{L}^3 + \int_{\partial_N \Omega} \overset{\circ}{\mathbf{c}} \cdot \mathbf{u}\, \mathrm{d}\mathcal{H}^2.$$

Note that if $\overset{\circ}{\mathbf{T}}$ is a residual stress then $\overset{\circ}{\mathsf{L}}[\mathbf{u}] = 0$. Let ε be a small real number. Suppose a small contact force $\varepsilon\mathbf{c} : \partial_N\Omega \to \mathbb{R}^3$ be superposed to $\overset{\circ}{\mathbf{c}}$ on $\partial_N\Omega$, and that a small distance force $\varepsilon\mathbf{d} : \Omega \to \mathbb{R}^3$ be superposed to $\overset{\circ}{\mathbf{d}}$ on Ω. Then, the *perturbed energy* is

$$\mathsf{E}(\varepsilon; \mathbf{u}) = \int_{\Omega} \mathsf{w}(\cdot, \mathbf{E}(\mathbf{u}))\, \mathrm{d}\mathcal{L}^3 - \overset{\circ}{\mathsf{L}}[\mathbf{u}] - \varepsilon\mathsf{L}[\mathbf{u}],$$

where

$$\mathsf{L}[\mathbf{u}] = \int_{\Omega} \mathbf{d} \cdot \mathbf{u}\, \mathrm{d}\mathcal{L}^3 + \int_{\partial_N \Omega} \mathbf{c} \cdot \mathbf{u}\, \mathrm{d}\mathcal{H}^2.$$

To capture the limit behavior of the displacement as ε approaches zero, we perform the substitution $\mathbf{u} \mapsto \varepsilon\mathbf{u}$. Using the Taylor expansion:

$$\mathsf{w}(x, \mathbf{E}) = \mathsf{w}(x, \mathbf{0}) + \partial\mathsf{w}(x, \mathbf{0}) \cdot \mathbf{E} + \frac{1}{2}\partial^2\mathsf{w}(x, \mathbf{0})[\mathbf{E}] \cdot \mathbf{E} + o(|\mathbf{E}|^2),$$

and taking into consideration (1) and (2), we obtain

$$\mathsf{E}(\varepsilon; \varepsilon\mathbf{u}) - \mathsf{E}(0; \mathbf{0}) = \varepsilon^2(\mathsf{Q}(\mathbf{u}) - \mathsf{L}[\mathbf{u}]) + o(\varepsilon^2), \tag{3}$$

where

$$\mathsf{Q}(\mathbf{u}) = \int_{\Omega} \frac{1}{2}\overset{\circ}{\mathbf{T}} \cdot \nabla\mathbf{u}^{\mathsf{T}}\nabla\mathbf{u} + \frac{1}{2}\mathbb{L}[\mathbf{e}(\mathbf{u})] \cdot \mathbf{e}(\mathbf{u})\, \mathrm{d}\mathcal{L}^3, \tag{4}$$

with $\mathbf{e}(\mathbf{u}) = \mathrm{sym}\nabla\mathbf{u}$, and

$$\mathbb{L}(x) = \partial^2\mathsf{w}(x, \mathbf{0}). \tag{5}$$

Thus from (3) we find

$$\lim_{\varepsilon\to 0} \frac{\mathsf{E}(\varepsilon; \varepsilon\mathbf{u}) - \mathsf{E}(0; \mathbf{0})}{\varepsilon^2} = \mathsf{Q}(\mathbf{u}) - \mathsf{L}[\mathbf{u}], \tag{6}$$

which holds for every smooth enough $\mathbf{u}$. The formal calculation leading to (6), and similar ones, motivates the replacement of the non-linear energy with its quadratic approximation whenever the loads are small. However, (6) alone does not justify this simplification: in addition, one needs to prove that a sequence of minimizers (or, more generally, quasi-minimizers) of

$$\frac{\mathsf{E}(\varepsilon; \varepsilon\cdot) - \mathsf{E}(0; \mathbf{0})}{\varepsilon^2}, \tag{7}$$

is close, in an appropriate norm and for small ε, to the minimizer of $\mathsf{Q}(\cdot) - \mathsf{L}[\cdot]$.

The aim of the present paper is to show that (almost) minimizers of (7) converge, in an appropriate sense, as ε approaches zero, to the minimizer of $\mathsf{Q}(\cdot)-\mathsf{L}[\cdot]$. This will be achieved by first proving that the sequence of functionals in (7) Γ-converges to $\mathsf{Q}(\cdot)-\mathsf{L}[\cdot]$ in an appropriate topology, and then by proving equi-compactness of the family of functionals in (7) with respect to the same topology.

Dal Maso et al. [2] studied the same problem under the requirement that $\mathsf{w}(x,\mathbf{E})$ have a minimum at $\mathbf{E}=\mathbf{0}$, which by (2) entails a null initial stress. A similar problem has been studied by Schmidt [15] for multi-well energies. There are two main differences between their framework and ours, which forced us to come up with a new proof of the Γ-converge result. The first, the presence of an initial stress entails the lack of convexity of the function (*cf.* (4)):

$$\mathbf{H}\mapsto\frac{1}{2}\overset{\circ}{\mathbf{T}}(x)\cdot\mathbf{H}^{\mathsf{T}}\mathbf{H}+\frac{1}{2}\mathbb{L}(x)[\mathrm{sym}\mathbf{H}]\cdot\mathrm{sym}\mathbf{H}$$

which might be concave in some directions for some values of $\overset{\circ}{\mathbf{T}}(x)$; for instance, for $\overset{\circ}{\mathbf{T}}(x)=-\mathbf{I}$ and $\mathbf{H}=s\mathbf{W}$ skew-symmetric the function above is equal to $-s^2|\mathbf{W}|^2/2$. The second, which is even more troublesome, the sequence of integrands of the elastic energy of the functionals (7) are unbounded below. This follows simply because the assumption of non-null initial stress implies that the function $\mathsf{w}(x,\cdot)-\mathsf{w}(x,\mathbf{0})$ takes negative values at least near $\mathbf{0}$, and hence $[\mathsf{w}(x,\cdot)-\mathsf{w}(x,\mathbf{0})]/\varepsilon^2$ has $-\infty$ limit as ε approaches zero. With regard to this second point, our argument was partly inspired by a recent work of Krömer [9].

In closing this introduction, we remark that our results are a substantial improvement over [14], where the problem we consider here was solved for uniform bodies whose stored energy density is a quadratic function of $\mathbf{E}$. In addition, the proof given here is completely different from that given in [14], which exploited a special structure of the energy density.

2 $\boldsymbol{\Gamma}$-convergence with initial stress

For the reader's convenience, we repeat here some of the notation already explained in the introduction. We let $\Omega\subset\mathbb{R}^3$ be an open connected domain with Lipschitz boundary $\partial\Omega$. We take Ω as the reference configuration of a hyperelastic body clamped on a part of the boundary $\partial_D\Omega$ with positive area, i.e., $\mathcal{H}^2(\partial_D\Omega)>0$. We let $\partial_N\Omega:=\partial\Omega\backslash\partial_D\Omega$. We denote by $\mathbf{u}:\Omega\to\mathbb{R}^3$ the displacement field, by $\nabla\mathbf{u}$ the displacement gradient, and by

$$\mathbf{E}(\nabla\mathbf{u})=\mathrm{sym}\nabla\mathbf{u}+\frac{1}{2}\nabla\mathbf{u}^{\mathsf{T}}\nabla\mathbf{u}\tag{8}$$

the Green–Saint Venant strain tensor.

Let $\mathbb{R}^{3\times3}_{\mathrm{Sym}}$ be the space of symmetric 3×3 matrices. We use $\bar{\mathsf{w}}:\Omega\times\mathbb{R}^{3\times3}\to\mathbb{R}\cup\{+\infty\}$ and $\mathsf{w}:\Omega\times\mathbb{R}^{3\times3}_{\mathrm{Sym}}\to\mathbb{R}\cup\{+\infty\}$ to denote the same stored energy density of the body, respectively, as a function of the deformation gradient $\mathbf{F}=\mathbf{I}+\nabla\mathbf{u}$ and of the Green–Saint Venant tensor, so that

$$\bar{\mathsf{w}}(\cdot,\mathbf{I}+\nabla\mathbf{u})=\mathsf{w}(\cdot,\mathbf{E}(\nabla\mathbf{u}))\tag{9}$$

for every smooth function $\mathbf{u}$. Both representation are needed for our purposes. The first, as we shall see, appears to be better suited to writing conditions in a neighborhood of $\nabla\mathbf{u}=\mathbf{0}$, needed for Γ-convergence (*cf.* assumptions (H1–H3) below). The second, on the other hand, automatically incorporates the requirement of frame indifference, and appears to be better suited for assumptions involving the behavior of the energy at large strains (*cf.* assumptions (H5–H6) in Sect. 3).

The hypotheses we make on $\bar{\mathsf{w}}$ are the following:

(H1) $\bar{\mathsf{w}}$ is a Carathéodory function on $\Omega\times\mathbb{R}^{3\times3}_+$ (by $\mathbb{R}^{3\times3}_+$, we denote the set of 3×3 matrices with strictly positive determinant), i.e., $\bar{\mathsf{w}}(\cdot,\mathbf{F})$ is measurable on Ω for every $\mathbf{F}\in\mathbb{R}^{3\times3}_+$, and $\bar{\mathsf{w}}(x,\cdot)$ is continuous on $\mathbb{R}^{3\times3}_+$ for almost every $x\in\Omega$. Moreover,

$$\bar{\mathsf{w}}(\cdot,\mathbf{F})=+\infty\ \text{ if }\det\mathbf{F}\le0\ \text{ and }\bar{\mathsf{w}}(\cdot,\mathbf{F})\to+\infty\ \text{ if }\det\mathbf{F}\searrow0,$$

almost everywhere in Ω.

(H2) The function $\mathbf{F} \mapsto \bar{\mathsf{w}}(x, \mathbf{F})$ is differentiable at $\mathbf{F} = \mathbf{I}$ for a.e. $x \in \Omega$, and the initial stress $\overset{\circ}{\mathbf{T}} := \partial\bar{\mathsf{w}}(\cdot, \mathbf{I}) \in L^\infty(\Omega; \mathbb{R}^{3\times3}_{\mathrm{Sym}})$.

(H3) There exists $\delta > 0$ such that the function $(x, \mathbf{F}) \mapsto \bar{\mathsf{w}}(x, \mathbf{F})$ is twice differentiable with respect to $\mathbf{F}$ on $\Omega \times \{\mathbf{F} \in \mathbb{R}^{3\times3} : \mathrm{dist}\,(\mathbf{F}, SO(3)) \le \delta\}$. Moreover $\partial^2\bar{\mathsf{w}}(\cdot, \mathbf{I}) \in L^\infty(\Omega)$ and there exist a non-negative $g \in L^\infty(\Omega)$ and a continuous function $\beta : [0, \delta) \to [0, +\infty)$ with $\beta(0) = 0$ such that

$$|\partial^2\bar{\mathsf{w}}(x, \mathbf{I} + \mathbf{H}) - \partial^2\bar{\mathsf{w}}(x, \mathbf{I})| \le g(x)\beta(|\mathbf{H}|),$$

for a.e. $x \in \Omega$ and for every $|\mathbf{H}| \le \delta$.

Before proceeding further on, we point out the following immediate consequences of (H1–H3), (8), and (9): (i) w is a Caratheodory's function; (ii) the function $\mathbf{E} \mapsto \mathsf{w}(x, \mathbf{E})$ is twice continuously differentiable near the origin; (iii) the residual stress $\overset{\circ}{\mathbf{T}}$ is given by

$$(\overset{\circ}{\mathbf{T}}(\cdot) :=)\quad \partial\bar{\mathsf{w}}(\cdot, \mathbf{I}) = \partial\mathsf{w}(\cdot, \mathbf{0}); \tag{10}$$

(iv) for every $\mathbf{H} \in \mathbb{R}^{3\times3}$,

$$\mathbb{A}(\cdot)[\mathbf{H}] \cdot \mathbf{H} = \overset{\circ}{\mathbf{T}}(\cdot) \cdot \mathbf{H}^{\mathsf{T}}\mathbf{H} + \mathbb{L}(\cdot)[\mathrm{sym}\mathbf{H}] \cdot \mathrm{sym}\mathbf{H}, \tag{11}$$

where

$$\mathbb{A}(\cdot) := \partial^2\bar{\mathsf{w}}(\cdot, \mathbf{I}), \quad \text{and} \quad \mathbb{L}(\cdot) := \partial^2\mathsf{w}(\cdot, \mathbf{0}),$$

cf. [7,12] and [11].

In particular, (11) implies that definition (4) can be equivalently written as:

$$\mathsf{Q}(\mathbf{u}) = \int_\Omega \frac{1}{2}\mathbb{A}[\nabla\mathbf{u}] \cdot \nabla\mathbf{u}\, \mathrm{d}\mathcal{L}^3. \tag{12}$$

The identity (10) is readily established. Let us prove the identity (11), which is less immediate. Indeed, defining the map $\hat{\mathbf{E}} : \mathbf{H} \mapsto \mathrm{sym}\mathbf{H} + \frac{1}{2}\mathbf{H}^{\mathsf{T}}\mathbf{H}$, differentiating with respect to $\mathbf{H}$ the identity $\bar{\mathsf{w}}(\cdot, \mathbf{I} + \mathbf{H}) = \mathsf{w}(\cdot, \hat{\mathbf{E}}(\mathbf{H}))$, and using the chain rule, we obtain

$$\begin{aligned}
\partial\bar{\mathsf{w}}(\cdot, \mathbf{I} + \mathbf{H}) \cdot \mathbf{A} &= \partial\mathsf{w}(\cdot, \mathbf{E}(\mathbf{H})) \cdot \partial\hat{\mathbf{E}}(\mathbf{H})[\mathbf{A}] \\
&= \partial\mathsf{w}(\cdot, \mathbf{E}(\mathbf{H})) \cdot \left(\mathrm{sym}\mathbf{A} + \frac{1}{2}\mathbf{H}^{\mathsf{T}}\mathbf{A} + \frac{1}{2}\mathbf{A}^{\mathsf{T}}\mathbf{H}\right) \\
&= \partial\mathsf{w}(\cdot, \mathbf{E}(\mathbf{H})) \cdot \mathrm{sym}\left(\mathbf{A} + \mathbf{H}^{\mathsf{T}}\mathbf{A}\right),
\end{aligned} \tag{13}$$

where we have used the symmetry of $\partial\mathsf{w}(\cdot, \mathbf{E})$. Thus, by a further differentiation with respect to $\mathbf{H}$, and by setting $\mathbf{H} = \mathbf{0}$, we eventually obtain:

$$\partial^2\bar{\mathsf{w}}(\cdot, \mathbf{I})[\mathbf{B}] \cdot \mathbf{A} = \partial\mathsf{w}(\cdot, \mathbf{0}) \cdot \mathbf{B}^{\mathsf{T}}\mathbf{A} + \partial^2\mathsf{w}(\cdot, \mathbf{0})[\mathrm{sym}\mathbf{B}] \cdot \mathrm{sym}\mathbf{A}.$$

Recalling that $\partial^2\bar{\mathsf{w}}(\cdot, \mathbf{I}) = \mathbb{A}(\cdot)$, $\partial\mathsf{w}(\cdot, \mathbf{0}) = \overset{\circ}{\mathbf{T}}(\cdot)$, and $\partial^2\mathsf{w}(\cdot, \mathbf{0}) = \mathbb{L}(\cdot)$, we obtain (11).

The reference configuration is subject to a system of non-vanishing dead loads consisting in a system of distance forces $\overset{\circ}{\mathbf{d}} + \varepsilon\mathbf{d} : \Omega \to \mathbb{R}^3$, and in a system of contact forces $\overset{\circ}{\mathbf{c}} + \varepsilon\mathbf{c} : \partial_N\Omega \to \mathbb{R}^3$, where $\varepsilon > 0$. We assume $\overset{\circ}{\mathbf{d}}, \mathbf{d} \in L^2(\Omega; \mathbb{R}^3)$ and $\overset{\circ}{\mathbf{c}}, \mathbf{c} \in L^2(\partial_N\Omega; \mathbb{R}^3)$. Let

$$H^1_{0_D}(\Omega; \mathbb{R}^3) := \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{v} = \mathbf{0} \text{ on } \partial_D\Omega\}.$$

Then, the total energy of the system $\mathsf{E}(\varepsilon; \cdot) : H^1_{0_D}(\Omega; \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ is

$$\mathsf{E}(\varepsilon; \mathbf{u}) = \int_\Omega \bar{\mathsf{w}}(\cdot, \mathbf{I} + \nabla\mathbf{u})\, \mathrm{d}\mathcal{L}^3 - \overset{\circ}{\mathsf{L}}[\mathbf{u}] - \varepsilon\mathsf{L}[\mathbf{u}],$$

where $\overset{\circ}{\mathsf{L}}$ and L are defined as in the introduction, i.e.,

$$\overset{\circ}{\mathsf{L}}[\mathbf{u}] = \int_{\Omega} \overset{\circ}{\mathbf{d}} \cdot \mathbf{u} \, \mathrm{d}\mathcal{L}^3 + \int_{\partial_N \Omega} \overset{\circ}{\mathbf{c}} \cdot \mathbf{u} \, \mathrm{d}\mathcal{H}^2, \qquad \mathsf{L}[\mathbf{u}] = \int_{\Omega} \mathbf{d} \cdot \mathbf{u} \, \mathrm{d}\mathcal{L}^3 + \int_{\partial_N \Omega} \mathbf{c} \cdot \mathbf{u} \, \mathrm{d}\mathcal{H}^2.$$

In order for the linearization to be meaningful, we require that the reference configuration be equilibrated and infinitesimally stable (*cf.* [17, §69bis]), more precisely that the initial stress $\overset{\circ}{\mathbf{T}}$ equilibrates the loads $(\overset{\circ}{\mathbf{d}}, \overset{\circ}{\mathbf{c}})$ and that the functional Q given by (12) is non-negative. We note that, by (H3), $|\mathbb{A}| \in L^{\infty}(\Omega)$.

(H4) The residual stress $\overset{\circ}{\mathbf{T}}$ and the fourth- order tensor $\mathbb{A}$ satisfy

$$\int_{\Omega} \overset{\circ}{\mathbf{T}} \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}^3 = \int_{\Omega} \overset{\circ}{\mathbf{b}} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}^3 + \int_{\partial_N \Omega} \overset{\circ}{\mathbf{c}} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{H}^2, \tag{14}$$

and

$$\int_{\Omega} \mathbb{A}[\nabla \boldsymbol{\varphi}] \cdot \nabla \boldsymbol{\varphi}, \mathrm{d}\mathcal{L}^3 \geq 0, \tag{15}$$

for every $\boldsymbol{\varphi} \in H^1_{0_D}(\Omega; \mathbb{R}^3)$.

We now show that if the null displacement is an H^1-local minimizer of the energy $\mathsf{E}(0; \cdot)$, then (H4) holds.

Lemma 1 *Assume* (H1–H3). *Suppose that there exists an* $\eta > 0$ *such that*

$$\mathsf{E}(0; \mathbf{0}) \leq \mathsf{E}(0; \boldsymbol{\varphi}), \tag{16}$$

for every $\boldsymbol{\varphi} \in H^1_{0_D}(\Omega; \mathbb{R}^3)$ *with* $\|\boldsymbol{\varphi}\|_{H^1(\Omega)} \leq \eta$. *Then* (H4) *holds true.*

Proof It suffices to prove the Lemma for any $\boldsymbol{\varphi} \in C^{\infty}(\Omega; \mathbb{R}^3)$ such that $\boldsymbol{\varphi} = \mathbf{0}$ near $\partial_D \Omega$. The general case then follows by a standard density argument. Let $s > 0$. By Taylor's theorem,

$$\bar{\mathsf{w}}(\cdot, \mathbf{I} + s\nabla \boldsymbol{\varphi}) = \bar{\mathsf{w}}(\cdot, \mathbf{I}) + s\partial \bar{\mathsf{w}}(\cdot, \mathbf{I}) \cdot \nabla \boldsymbol{\varphi} + \frac{1}{2} s^2 \partial^2 \bar{\mathsf{w}}(\cdot, \mathbf{I} + s\xi_s \nabla \boldsymbol{\varphi})[\nabla \boldsymbol{\varphi}] \cdot \nabla \boldsymbol{\varphi},$$

with $\xi_s(x) \in (0, 1)$ for *a.e.* $x \in \Omega$. From (H3), for s small enough, we have that

$$|\partial^2 \bar{\mathsf{w}}(\cdot, \mathbf{I} + s\xi_s \nabla \boldsymbol{\varphi})[\nabla \boldsymbol{\varphi}] \cdot \nabla \boldsymbol{\varphi}| \leq K |\nabla \boldsymbol{\varphi}|^2 \quad \text{a.e. in } \Omega. \tag{17}$$

By (16), we then have

$$\begin{aligned} 0 \leq \frac{\mathsf{E}(0; s\boldsymbol{\varphi}) - \mathsf{E}(0; \mathbf{0})}{s} &= \int_{\Omega} \overset{\circ}{\mathbf{T}} \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}^3 - \int_{\Omega} \overset{\circ}{\mathbf{b}} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}^3 - \int_{\partial_N \Omega} \overset{\circ}{\mathbf{c}} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{H}^2 \\ &+ s \int_{\Omega} \frac{1}{2} \partial^2 \bar{\mathsf{w}}(\cdot, \mathbf{I} + s\xi_s \nabla \boldsymbol{\varphi})[\nabla \boldsymbol{\varphi}] \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}^3, \end{aligned} \tag{18}$$

hence, letting s go to zero we deduce, using (17) and the dominated convergence theorem, that

$$\int_{\Omega} \overset{\circ}{\mathbf{T}} \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}^3 - \int_{\Omega} \overset{\circ}{\mathbf{b}} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}^3 - \int_{\partial_N \Omega} \overset{\circ}{\mathbf{c}} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{H}^2 \geq 0,$$

from which (14) follows. Finally, from (18), taking into account (14) and using again (17) and the dominated convergence theorem, we obtain (15). ◻

Remark 1 If the initial stress $\overset{\circ}{\mathbf{T}}=\mathbf{0}$ then the reference configuration is said to be a natural configuration. In this case, from the previous Lemma, it follows that $(\overset{\circ}{\mathbf{d}},\overset{\circ}{\mathbf{c}})=(\mathbf{0},\mathbf{0})$. If instead $(\overset{\circ}{\mathbf{d}},\overset{\circ}{\mathbf{c}})=(\mathbf{0},\mathbf{0})$ and $\overset{\circ}{\mathbf{T}}\neq\mathbf{0}$ then $\overset{\circ}{\mathbf{T}}$ is said to be a residual stress.

We are now ready to state the main result of this section.

Theorem 1 *Assume* (H1–H4). *Then, the sequence of functionals* $(\mathsf{E}(\varepsilon;\varepsilon\cdot)-\mathsf{E}(0;\mathbf{0}))/\varepsilon^2$ *sequentially Gamma-converges, with respect to the weak-*$H^1_{0_D}(\Omega;\mathbb{R}^3)$ *convergence, to the functional:*

$$H^1_{0_D}(\Omega;\mathbb{R}^3)\ni\mathbf{u}\mapsto\mathsf{Q}(\mathbf{u})-\mathsf{L}[\mathbf{u}]. \tag{19}$$

That is

(i) *(liminf inequality) for every sequence* $\{\mathbf{u}_\varepsilon\}\subset H^1_{0_D}(\Omega;\mathbb{R}^3)$, *and every* $\mathbf{u}\in H^1_{0_D}(\Omega;\mathbb{R}^3)$ *such that* $\mathbf{u}_\varepsilon\rightharpoonup\mathbf{u}$ *in* $H^1(\Omega;\mathbb{R}^3)$ *we have that*

$$\liminf_{\varepsilon\to0}\frac{\mathsf{E}(\varepsilon;\varepsilon\mathbf{u}_\varepsilon)-\mathsf{E}(0;\mathbf{0})}{\varepsilon^2}\geq\mathsf{Q}(\mathbf{u})-\mathsf{L}[\mathbf{u}];$$

(ii) *(recovery sequence) for every* $\mathbf{u}\in H^1_{0_D}(\Omega;\mathbb{R}^3)$ *there exists a sequence* $\{\mathbf{u}_\varepsilon\}\subset H^1_{0_D}(\Omega;\mathbb{R}^3)$ *such that* $\mathbf{u}_\varepsilon\rightharpoonup\mathbf{u}$ *in* $H^1(\Omega;\mathbb{R}^3)$ *and*

$$\limsup_{\varepsilon\to0}\frac{\mathsf{E}(\varepsilon;\varepsilon\mathbf{u}_\varepsilon)-\mathsf{E}(0;\mathbf{0})}{\varepsilon^2}\leq\mathsf{Q}(\mathbf{u})-\mathsf{L}[\mathbf{u}].$$

Before proving Theorem 1, we state and prove some technical lemmas. In the body of a proof, C denotes a constant which may vary from one formula to another.

Lemma 2 *(decomposition lemma). Let* $\Omega\subset\mathbb{R}^3$ *be an open, bounded set with Lipschitz boundary and let* $\{\mathbf{u}_n\}$ *be a bounded sequence in* $H^1(\Omega;\mathbb{R}^3)$. *There exists a subsequence,* $\{\mathbf{u}_{n_k}\}$ *and a sequence* $\{\mathbf{z}_k\}\subset W^{1,\infty}(\Omega;\mathbb{R}^3)$ *such that:*

1. $\mathcal{L}^3(\{\mathbf{z}_k\neq\mathbf{u}_{n_k}\}\cup\{\nabla\mathbf{z}_k\neq\nabla\mathbf{u}_{n_k}\})\to0$;
2. $\{|\nabla\mathbf{z}_k|^2\}$ *is equi-integrable.*

Moreover, if $\mathbf{u}_n=\mathbf{0}$ *on* $\partial_D\Omega\subset\partial\Omega$ *with* $\mathcal{H}^2(\partial_D\Omega)\neq0$, *and if* $\{\lambda_k\}\subset\mathbb{R}$ *is any sequence such that* $\lambda_k\nearrow+\infty$, *the sequence* $\{\mathbf{z}_k\}\subset W^{1,\infty}(\Omega;\mathbb{R}^3)$ *may be chosen so that* $\mathbf{z}_k=\mathbf{0}$ *on* $\partial_D\Omega$, *and*

$$\|\nabla\mathbf{z}_k\|_{L^\infty}\leq C\lambda_k,$$

where C *does not depend on either* $\{\mathbf{u}_n\}$ *or* $\{\lambda_k\}$.

Proof The statement of the Lemma up to the "Moreover" part is a particular case of Lemma 1.2 of [5]. We now prove the "Moreover" part. We begin our proof with an adaptation of the proof of Lemma 1.2 of [5]. For brevity, we sketch it and we refer to [5] for further details.

Step 1 *(Boundary conditions)* Take $\eta_k\in C^\infty(\overline{\Omega})$ satisfying $0\leq\eta_k\leq1$ and

$$\eta_k(x)=\begin{cases}1 & \text{if } \operatorname{dist}(x,\partial_D\Omega)\leq1/k,\\ 0 & \text{if } \operatorname{dist}(x,\partial_D\Omega)\geq2/k.\end{cases}$$

By the density of

$$C^\infty_{0_D}(\Omega;\mathbb{R}^3):=\{\mathbf{v}\in C^\infty(\Omega):\operatorname{dist}(\{\mathbf{v}\neq0\},\partial_D\Omega)>0\}$$

in $H^1_{0_D}(\Omega;\mathbb{R}^3)$, there exists $\mathbf{v}_k\in C^\infty_{0_D}(\Omega;\mathbb{R}^3)$ such that

$$\|\mathbf{v}_k-\mathbf{u}_k\|_{H^1(\Omega)}\leq\min\left(1,\|\nabla\eta_k\|^{-1}_{L^\infty(\Omega)}\right). \tag{20}$$

We define

$$\widetilde{\mathbf{z}}_k := \eta_k \mathbf{v}_k + (1 - \eta_k)\mathbf{u}_k.$$

Then,

$$\{\widetilde{\mathbf{z}}_k \neq \mathbf{u}_k\} \subset \Sigma_k := \{x \in \Omega : \mathrm{dist}(x, \partial_D \Omega) \leq 2/k\}. \tag{21}$$

Moreover,

$$\{\widetilde{\mathbf{z}}_k = \mathbf{0}\} \supset S_k := \{x \in \Omega : \mathrm{dist}(x, \partial_D \Omega) \leq \min(1/k, d_k)\}, \tag{22}$$

where

$$d_k := \mathrm{dist}(\{\mathbf{v}_k \neq \mathbf{0}\}, \partial_D \Omega) > 0.$$

Furthermore,

$$\begin{aligned}\|\widetilde{\mathbf{z}}_k\|^2_{H^1(\Omega)} &\leq 2\|\mathbf{u}_k\|^2_{H^1(\Omega)} + 2\|\eta_k(\mathbf{v}_k - \mathbf{u}_k)\|^2_{H^1(\Omega)} \\ &\leq 2\|\mathbf{u}_k\|^2_{H^1(\Omega)} + 2\|\nabla \eta_k\|^2_{L^\infty(\Omega)}\|\mathbf{v}_k - \mathbf{u}_k\|^2_{L^2(\Omega)} + 2\|\mathbf{v}_k - \mathbf{u}_k\|^2_{H^1(\Omega)}.\end{aligned}$$

Hence, by (20),

$$\|\widetilde{\mathbf{z}}_k\|_{H^1(\Omega)} \leq C. \tag{23}$$

Step 2 *(Selection of the subsequence).* We introduce the truncations

$$T_k(x) = \begin{cases} x, & |x| \leq \lambda_k, \\ \lambda_k \frac{x}{|x|}, & |x| > \lambda_k. \end{cases} \tag{24}$$

and we identify $\widetilde{\mathbf{z}}_n$ with its extension to $\mathbb{R}^3$ obtained through a bounded extension operator from $H^1(\Omega; \mathbb{R}^3)$ to $H^1(\mathbb{R}^3; \mathbb{R}^3)$. Then,

$$\|\widetilde{\mathbf{z}}_n\|_{H^1(\mathbb{R}^3)} \leq C. \tag{25}$$

Next, we consider the *maximal function* of $\nabla \widetilde{\mathbf{z}}_n$ by setting, for every $x \in \mathbb{R}^3$,

$$M(\nabla \widetilde{\mathbf{z}}_n)(x) = \sup_{r>0} \frac{1}{\mathcal{L}^3(B(x,r))} \int_{B(x,r)} |\nabla \widetilde{\mathbf{z}}_n| \, \mathrm{d}\mathcal{L}^3.$$

It is immediate that

$$|\nabla \widetilde{\mathbf{z}}_n| \leq M(\nabla \widetilde{\mathbf{z}}_n) \text{ a.e. in } \mathbb{R}^3. \tag{26}$$

It is also known that

$$\|M(\nabla \widetilde{\mathbf{z}}_n)\|_{L^2(\mathbb{R}^3)} \leq C\|\nabla \widetilde{\mathbf{z}}_n\|_{L^2(\mathbb{R}^3)}, \tag{27}$$

and that there exists a constant $C > 0$ such that for every $\lambda > 0$ (*cf.* [4, §6.6.3], Theorem 3)

$$|\nabla \widetilde{\mathbf{z}}_n| \leq C\lambda \text{ a.e. in } \{x \in \mathbb{R}^3 : M(\nabla \widetilde{\mathbf{z}}_n)(x) < \lambda\}. \tag{28}$$

By (25) and (27), the sequence $\{|M(\nabla \widetilde{\mathbf{z}}_n)|^2\}$ is bounded in $L^1(\Omega)$, and the argument based on Young measures contained in [5] allows us to extract a subsequence $\{\widetilde{\mathbf{z}}_{n_k}\}$ such that (*cf.* (4.5) of [5]):

$$|T_k(M(\nabla \widetilde{\mathbf{z}}_{n_k}))|^2 \rightharpoonup f \quad \text{in } L^1(\Omega), \tag{29}$$

for some $f \in L^1(\Omega)$.
Step 2 *(Truncation).* Let

$$R_k := \{x \in \mathbb{R}^3 : M(\nabla \widetilde{\mathbf{z}}_{n_k})(x) \geq \lambda_k\}. \tag{30}$$

By (25) and (27),

$$\mathcal{L}^3(R_k \cap \Omega) \leq \frac{1}{{\lambda_k}^2} \int_\Omega |M(\nabla \widetilde{\mathbf{z}}_{n_k})|^2 \, \mathrm{d}\mathcal{L}^3 \leq \frac{C}{\lambda_k^2} \to 0.$$

By (22), (28), and (30),

$$|\nabla \widetilde{\mathbf{z}}_{n_k}| \leq C\lambda_k \quad \text{a.e. in } (\Omega \backslash R_k) \cup S_k.$$

By a standard extension result (*cf.* [4, §3.1], Theorem 1), we obtain an extension $\mathbf{z}_k \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ of $\widetilde{\mathbf{z}}_k$ such that

$$\mathbf{z}_k = \widetilde{\mathbf{z}}_{n_k} \text{ in } (\Omega \backslash R_k) \cup S_k, \tag{31}$$

and

$$||\nabla \mathbf{z}_k||_{L^\infty(\Omega)} \leq C\lambda_k. \tag{32}$$

It follows from (22) and (31) that $\operatorname{dist}(\{\mathbf{z}_k \neq \mathbf{0}\}, \partial_D \Omega) > 0$, hence

$$\mathbf{z}_k = \mathbf{0} \text{ in } \partial_D \Omega,$$

in the sense of traces. Moreover, since $\mathbf{z}_k = \widetilde{\mathbf{z}}_{n_k} = \mathbf{u}_{n_k}$ in $\Omega \backslash (\Sigma_k \cup R_k)$, we have

$$\mathcal{L}^3(\{\mathbf{z}_k \neq \mathbf{u}_{n_k}\} \cup \{\nabla \mathbf{z}_k \neq \nabla \mathbf{u}_{n_k}\}) \leq \mathcal{L}^3(\Sigma_k \cup R_k) \to 0.$$

By (32) and (30), we have

$$|\nabla \mathbf{z}_k| \leq C\lambda_k \leq C T_k(M(\nabla \widetilde{\mathbf{z}}_{n_k})) \text{ a.e. in } R_k.$$

Furthermore, by (31), (26), and (30), we have

$$|\nabla \mathbf{z}_k| = |\nabla \widetilde{\mathbf{z}}_{n_k}| \leq C M(\nabla \widetilde{\mathbf{z}}_{n_k}) = C T_k(M(\nabla \widetilde{\mathbf{z}}_{n_k})) \text{ a.e. in } \Omega \backslash R_k, \tag{33}$$

thus, by (29), the sequence $\{|\nabla \mathbf{z}_k|^2\}$ is equi-integrable.

The next lemma is proved in [13]. For the reader's convenience, we sketch a proof.

Lemma 3 *Assume that* $|\mathbb{A}| \in L^\infty(\Omega)$ *and* $\int_\Omega \mathbb{A}[\nabla \mathbf{z}] \cdot \nabla \mathbf{z} \, d\mathcal{L}^3 \geq 0$ *for all* $\mathbf{z}$ *in* $H^1_{0_D}(\Omega; \mathbb{R}^3)$*, then*

$$\liminf_{\varepsilon \to 0} \int_\Omega \mathbb{A}[\nabla \mathbf{z}_\varepsilon] \cdot \nabla \mathbf{z}_\varepsilon \, d\mathcal{L}^3 \geq \int_\Omega \mathbb{A}[\nabla \mathbf{z}] \cdot \nabla \mathbf{z} \, d\mathcal{L}^3 \tag{34}$$

whenever $\{\mathbf{z}_\varepsilon\}$ *converges to* $\mathbf{z}$ *weakly in* $H^1_{0_D}(\Omega; \mathbb{R}^{3\times 3})$.

Proof By assumption, $\int_\Omega \mathbb{A}[\nabla \mathbf{z}_\varepsilon - \nabla \mathbf{z}] \cdot (\nabla \mathbf{z}_\varepsilon - \nabla \mathbf{z}) \, \mathrm{d}\mathcal{L}^3 \geq 0$, hence

$$\int_\Omega \mathbb{A}[\nabla \mathbf{z}_\varepsilon] \cdot \nabla \mathbf{z}_\varepsilon \, \mathrm{d}\mathcal{L}^3 \geq \int_\Omega \Big(\mathbb{A}^\mathsf{T}[\nabla \mathbf{z}] + \mathbb{A}[\nabla \mathbf{z}]\Big) \cdot \nabla \mathbf{z}_\varepsilon - \mathbb{A}[\nabla \mathbf{z}] \cdot \nabla \mathbf{z} \, \mathrm{d}\mathcal{L}^3,$$

where $\mathbb{A}^\mathsf{T}$ denotes the fourth-order tensor defined by $\mathbb{A}^\mathsf{T}[\mathbf{A}] \cdot \mathbf{B} = \mathbb{A}[\mathbf{B}] \cdot \mathbf{A}$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}$. Since $\{\mathbf{z}_\varepsilon\}$ converges weakly, we have

$$\liminf_{\varepsilon \to 0} \int_\Omega \mathbb{A}[\nabla \mathbf{z}_\varepsilon] \cdot \nabla \mathbf{z}_\varepsilon \, \mathrm{d}\mathcal{L}^3 \geq \int_\Omega \mathbb{A}^\mathsf{T}[\nabla \mathbf{z}] \cdot \nabla \mathbf{z} \, \mathrm{d}\mathcal{L}^3 = \int_\Omega \mathbb{A}[\nabla \mathbf{z}] \cdot \nabla \mathbf{z} \, \mathrm{d}\mathcal{L}^3. \qquad \square$$

Proof of Theorem 1. We start by proving the liminf inequality. Let $\{\mathbf{u}_\varepsilon\}$ be a sequence in $H^1_{0_D}(\Omega;\mathbb{R}^3)$, $\mathbf{u} \in H^1_{0_D}(\Omega;\mathbb{R}^3)$ such that $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$ in $H^1(\Omega;\mathbb{R}^3)$. By (H4), we have

$$\mathring{\mathsf{L}}[\varepsilon\mathbf{u}_\varepsilon] = \int_\Omega \mathring{\mathbf{T}} \cdot (\varepsilon\nabla\mathbf{u}_\varepsilon)\,\mathrm{d}\mathcal{L}^3$$

and hence

$$\begin{aligned}\frac{\mathsf{E}(\varepsilon;\varepsilon\mathbf{u}_\varepsilon)-\mathsf{E}(0;\mathbf{0})}{\varepsilon^2} &= \frac{1}{\varepsilon^2}\left(\int_\Omega \bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\nabla\mathbf{u}_\varepsilon)-\bar{\mathsf{w}}(\cdot,\mathbf{I})\,\mathrm{d}\mathcal{L}^3 - \mathring{\mathsf{L}}[\varepsilon\mathbf{u}_\varepsilon]-\varepsilon\mathsf{L}[\varepsilon\mathbf{u}_\varepsilon]\right)\\ &= \frac{1}{\varepsilon^2}\left(\int_\Omega \bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\nabla\mathbf{u}_\varepsilon)-\bar{\mathsf{w}}(\cdot,\mathbf{I}) - \mathring{\mathbf{T}}\cdot(\varepsilon\nabla\mathbf{u}_\varepsilon)\,\mathrm{d}\mathcal{L}^3\right)-\mathsf{L}[\mathbf{u}_\varepsilon].\end{aligned}$$

Since $\lim_{\varepsilon\to 0}\mathsf{L}[\mathbf{u}_\varepsilon]=\mathsf{L}[\mathbf{u}]$ it suffices to prove that

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega \bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\nabla\mathbf{u}_\varepsilon)-\bar{\mathsf{w}}(\cdot,\mathbf{I}) - \mathring{\mathbf{T}}\cdot(\varepsilon\nabla\mathbf{u}_\varepsilon)\,\mathrm{d}\mathcal{L}^3 \ge \int_\Omega \frac{1}{2}\mathbb{A}[\nabla\mathbf{u}]\cdot\nabla\mathbf{u}\,\mathrm{d}\mathcal{L}^3. \tag{35}$$

By Lemma 2, there exists a sequence $\{\mathbf{z}_\varepsilon\}\subset H^1_{0_D}(\Omega;\mathbb{R}^3)$ such that $\{|\nabla\mathbf{z}_\varepsilon|^2\}$ is equi-integrable, $\|\nabla\mathbf{z}_\varepsilon\|_{L^\infty(\Omega)} \le 1/\sqrt{\varepsilon}$ and $\mathcal{L}^3(\mathcal{D}_\varepsilon)\to 0$ as $\varepsilon\to 0$, with $\mathcal{D}_\varepsilon := \{\mathbf{z}_\varepsilon \ne \mathbf{u}_\varepsilon\}\cup\{\nabla\mathbf{z}_\varepsilon \ne \nabla\mathbf{u}_\varepsilon\}$. Set $\mathbf{w}_\varepsilon := \mathbf{u}_\varepsilon - \mathbf{z}_\varepsilon$. Since $\|\varepsilon\nabla\mathbf{z}_\varepsilon\|_{L^\infty(\Omega)} \le \sqrt{\varepsilon}$ we have, for small ε, $\det(\mathbf{I}+\varepsilon\nabla\mathbf{z}_\varepsilon)>0$ and hence we may write

$$\frac{1}{\varepsilon^2}\int_\Omega \bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\nabla\mathbf{u}_\varepsilon)-\bar{\mathsf{w}}(\cdot,\mathbf{I}) - \mathring{\mathbf{T}}\cdot(\varepsilon\nabla\mathbf{u}_\varepsilon)\,\mathrm{d}\mathcal{L}^3 = B_\varepsilon + G_\varepsilon, \tag{36}$$

where

$$B_\varepsilon := \frac{1}{\varepsilon^2}\int_\Omega \bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\nabla\mathbf{z}_\varepsilon+\varepsilon\nabla\mathbf{w}_\varepsilon)-\bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\nabla\mathbf{z}_\varepsilon) - \mathring{\mathbf{T}}\cdot(\varepsilon\nabla\mathbf{w}_\varepsilon)\,\mathrm{d}\mathcal{L}^3,$$

and

$$G_\varepsilon := \frac{1}{\varepsilon^2}\int_\Omega \bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\nabla\mathbf{z}_\varepsilon)-\bar{\mathsf{w}}(\cdot,\mathbf{I}) - \mathring{\mathbf{T}}\cdot(\varepsilon\nabla\mathbf{z}_\varepsilon)\,\mathrm{d}\mathcal{L}^3.$$

We claim that

$$\liminf_{\varepsilon\to 0} B_\varepsilon \ge 0. \tag{37}$$

Note that by (H4), we have

$$\begin{aligned}B_\varepsilon &= \frac{1}{\varepsilon}\int_\Omega \partial\bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\nabla\mathbf{z}_\varepsilon)\cdot\nabla\mathbf{w}_\varepsilon+\frac{1}{2}\partial^2\bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\rho_\varepsilon\nabla\mathbf{z}_\varepsilon)[\varepsilon\nabla\mathbf{w}_\varepsilon]\cdot\nabla\mathbf{w}_\varepsilon - \mathring{\mathbf{T}}\cdot\nabla\mathbf{w}_\varepsilon\mathrm{d}\mathcal{L}^3\\ &= \frac{1}{\varepsilon}\int_\Omega [\partial\bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\nabla\mathbf{z}_\varepsilon)-\partial\bar{\mathsf{w}}(\cdot,\mathbf{I})]\cdot\nabla\mathbf{w}_\varepsilon\mathrm{d}\mathcal{L}^3\\ &\quad+\frac{1}{2}\int_\Omega \{\partial^2\bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\rho_\varepsilon\nabla\mathbf{z}_\varepsilon)-\partial^2\bar{\mathsf{w}}(\cdot,\mathbf{I})\}[\nabla\mathbf{w}_\varepsilon]\cdot\nabla\mathbf{w}_\varepsilon+\mathbb{A}[\nabla\mathbf{w}_\varepsilon]\cdot\nabla\mathbf{w}_\varepsilon\mathrm{d}\mathcal{L}^3\\ &\ge \int_\Omega \partial^2\bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\xi_\varepsilon\nabla\mathbf{z}_\varepsilon)[\nabla\mathbf{z}_\varepsilon]\cdot\nabla\mathbf{w}_\varepsilon\mathrm{d}\mathcal{L}^3\\ &\quad+\frac{1}{2}\int_\Omega \{\partial^2\bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\rho_\varepsilon\nabla\mathbf{z}_\varepsilon)-\partial^2\bar{\mathsf{w}}(\cdot,\mathbf{I})\}[\nabla\mathbf{w}_\varepsilon]\cdot\nabla\mathbf{w}_\varepsilon\mathrm{d}\mathcal{L}^3,\end{aligned} \tag{38}$$

where $0 < \rho_\varepsilon(x) < 1$ and $0 < \xi_\varepsilon(x) < 1$ for a.e. $x \in \Omega$. By (H3) and using that $\|\varepsilon\nabla\mathbf{z}_\varepsilon\|_{L^\infty(\Omega)} \le \sqrt{\varepsilon}$ and that $\mathbf{w}_\varepsilon = \mathbf{0}$ on $\Omega\backslash\mathcal{D}_\varepsilon$ we have

$$B_\varepsilon \ge -C\int_{\mathcal{D}_\varepsilon} |\nabla\mathbf{z}_\varepsilon|\,|\nabla\mathbf{w}_\varepsilon|\mathrm{d}\mathcal{L}^3 - \frac{1}{2}\|g\ \beta(|\varepsilon\rho_\varepsilon\nabla\mathbf{z}_\varepsilon|)\|_{L^\infty(\Omega)}\int_\Omega |\nabla\mathbf{w}_\varepsilon|^2\mathrm{d}\mathcal{L}^3$$
$$\ge -C(\int_\Omega |\nabla\mathbf{w}_\varepsilon|^2\mathrm{d}\mathcal{L}^3)^{1/2}(\int_{\mathcal{D}_\varepsilon} |\nabla\mathbf{z}_\varepsilon|^2\mathrm{d}\mathcal{L}^3)^{1/2}$$
$$-\frac{1}{2}\|g\ \beta(|\varepsilon\rho_\varepsilon\nabla\mathbf{z}_\varepsilon|)\|_{L^\infty(\Omega)}\int_\Omega |\nabla\mathbf{w}_\varepsilon|^2\mathrm{d}\mathcal{L}^3. \tag{39}$$

Inequality (37) now follows from (H3) and the equi-integrability of $\{|\nabla\mathbf{z}_\varepsilon|^2\}$.

We next study the limit of G_ε. By (H3), since $\|\varepsilon\nabla\mathbf{z}_\varepsilon\|_{L^\infty(\Omega)} \le \sqrt{\varepsilon}$, we have

$$G_\varepsilon = \frac{1}{\varepsilon^2}\int_\Omega \partial\bar{\mathsf{w}}(\cdot,\mathbf{I})\cdot\varepsilon\nabla\mathbf{z}_\varepsilon + \frac{1}{2}\partial^2\bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\rho_\varepsilon\nabla\mathbf{z}_\varepsilon)[\varepsilon\nabla\mathbf{z}_\varepsilon]\cdot\varepsilon\nabla\mathbf{z}_\varepsilon - \mathring{\mathbf{T}}\cdot(\varepsilon\nabla\mathbf{z}_\varepsilon)\mathrm{d}\mathcal{L}^3$$
$$= \int_\Omega \frac{1}{2}\partial^2\bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\rho_\varepsilon\nabla\mathbf{z}_\varepsilon)[\nabla\mathbf{z}_\varepsilon]\cdot\nabla\mathbf{z}_\varepsilon\,\mathrm{d}\mathcal{L}^3$$
$$= \int_\Omega \frac{1}{2}(\partial^2\bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\rho_\varepsilon\nabla\mathbf{z}_\varepsilon) - \mathbb{A})[\nabla\mathbf{z}_\varepsilon]\cdot\nabla\mathbf{z}_\varepsilon\,\mathrm{d}\mathcal{L}^3 + \int_\Omega \frac{1}{2}\mathbb{A}[\nabla\mathbf{z}_\varepsilon]\cdot\nabla\mathbf{z}_\varepsilon\,\mathrm{d}\mathcal{L}^3$$
$$\ge -\frac{1}{2}\|g\ \beta(|\varepsilon\rho_\varepsilon\nabla\mathbf{z}_\varepsilon|)\|_{L^\infty(\Omega)}\int_\Omega |\nabla\mathbf{z}_\varepsilon|^2\mathrm{d}\mathcal{L}^3 + \int_\Omega \frac{1}{2}\mathbb{A}[\nabla\mathbf{z}_\varepsilon]\cdot\nabla\mathbf{z}_\varepsilon\,\mathrm{d}\mathcal{L}^3$$

where $0 < \rho_\varepsilon(x) < 1$. Since $\mathbf{z}_\varepsilon \rightharpoonup \mathbf{u}$ in $H^1(\Omega;\mathbb{R}^3)$ and since $\|\varepsilon\nabla\mathbf{z}_\varepsilon\|_{L^\infty(\Omega)} \le \sqrt{\varepsilon}$ we have by Lemma 3 and (H3) that

$$\liminf_{\varepsilon\to 0} G_\varepsilon \ge \int_\Omega \frac{1}{2}\mathbb{A}[\nabla\mathbf{u}]\cdot\nabla\mathbf{u}\,\mathrm{d}\mathcal{L}^3. \tag{40}$$

The proof of the liminf inequality is then concluded.

We now prove the recovery sequence condition. Assume first that $\mathbf{u} \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ and $\mathbf{u} = \mathbf{0}$ in a neighborhood of $\partial_D\Omega$. Let $\mathbf{u}_\varepsilon := \mathbf{u}$. Then, taking into consideration (14), for ε small enough, we have

$$\frac{\mathsf{E}(\varepsilon;\varepsilon\mathbf{u}_\varepsilon) - \mathsf{E}(0;\mathbf{0})}{\varepsilon^2} = \frac{1}{\varepsilon^2}\left(\int_\Omega \bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\nabla\mathbf{u}) - \bar{\mathsf{w}}(\cdot,\mathbf{I})\,\mathrm{d}\mathcal{L}^3 - \varepsilon\,\mathring{\mathsf{L}}\,[\mathbf{u}] - \varepsilon^2\mathsf{L}[\mathbf{u}]\right)$$
$$= \int_\Omega \frac{1}{2}\partial^2\bar{\mathsf{w}}(\cdot,\mathbf{I}+\varepsilon\rho_\varepsilon\nabla\mathbf{u})\nabla\mathbf{u}\cdot\nabla\mathbf{u}\,\mathrm{d}\mathcal{L}^3 - \mathsf{L}[\mathbf{u}],$$

with $0 < \rho_\varepsilon(x) < 1$ for a.e. $x \in \Omega$. By (H3) and the dominated convergence theorem, we find

$$\lim_{\varepsilon\to 0}\frac{\mathsf{E}(\varepsilon;\varepsilon\mathbf{u}_\varepsilon) - \mathsf{E}(0;\mathbf{0})}{\varepsilon^2} = \mathsf{Q}(\mathbf{u}) - \mathsf{L}[\mathbf{u}].$$

Finally, we consider $\mathbf{u} \in H^1_{0_D}(\Omega;\mathbb{R}^3)$. Let $\mathbf{v}_k \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ and $\mathbf{u} = \mathbf{0}$ in a neighborhood of $\partial_D\Omega$ such that $\mathbf{v}_k \to \mathbf{u}$ in $H^1_{0_D}(\Omega;\mathbb{R}^3)$ as $k \to +\infty$. Then

$$\lim_{k\to+\infty}\lim_{\varepsilon\to 0}\frac{\mathsf{E}(\varepsilon;\varepsilon\mathbf{v}_k) - \mathsf{E}(0;\mathbf{0})}{\varepsilon^2} = \lim_{k\to+\infty}(\mathsf{Q}(\mathbf{v}_k) - \mathsf{L}[\mathbf{v}_k]) = \mathsf{Q}(\mathbf{u}) - \mathsf{L}[\mathbf{u}].$$

From a diagonalization argument, there exists an increasing function $\varepsilon \mapsto k(\varepsilon)$ such that

$$\lim_{\varepsilon \to 0} \frac{\mathsf{E}(\varepsilon; \varepsilon \mathbf{v}_{k(\varepsilon)}) - \mathsf{E}(0; \mathbf{0})}{\varepsilon^2} = \mathsf{Q}(\mathbf{u}) - \mathsf{L}[\mathbf{u}].$$

The proof is concluded by setting $\mathbf{u}_\varepsilon = \mathbf{v}_{k(\varepsilon)}$. □

3 Convergence of almost minimizers

The Γ convergence result of the previous section was obtained under the assumption that the reference configuration was only a local minimizer of the energy with null loads. Γ convergence is not well suited for studying the convergence of local minimizers, even if there are a few papers discussing the issue. The first is due to Kohn and Sternberg [8] where, under appropriate assumptions, the existence and the convergence of local minimizers for a sequence of functionals is determined (under the assumption that the limit functional has a local minimizer). These results apply to quasi-convex integrands with a p-growth from above; it is well known that this last assumption is incompatible with impenetrability of matter, i.e., our assumption (H1). Here, we prefer not to make unphysical assumptions on the energy density and to restrict ourselves to the discussion of the convergence of (almost) global minimizers.

To avoid the issue of existence of minimizers, we consider a ε-parameterized collection $\{\mathbf{u}_\varepsilon\}$ of almost minimizers:

$$\mathsf{E}(\varepsilon, \varepsilon \mathbf{u}_\varepsilon) = \inf_{\boldsymbol{\varphi} \in H^1_{0_D}(\Omega;\mathbb{R}^3)} \mathsf{E}(\varepsilon, \varepsilon \boldsymbol{\varphi}) + o(\varepsilon^2). \tag{41}$$

In this section, we show (*cf.* Theorem 2 below) that, as $\varepsilon \to 0$, almost minimizers converge, in the appropriate sense, to a solution of the incremental problem:

$$\min_{\boldsymbol{\varphi} \in H^1_{0_D}(\Omega;\mathbb{R}^3)} \left(\mathsf{Q}(\boldsymbol{\varphi}) - \mathsf{L}(\boldsymbol{\varphi})\right), \tag{42}$$

and that such solution is unique.

Before we address the convergence of minimizers, we shall give sufficient conditions for the existence of a unique solution to the incremental problem (42). By Lemma 3, the functional Q is lower-semicontinuous with respect to the weak-H^1 convergence. However, condition (15) is not enough to guarantee compactness in H^1. Consequently, the assumptions made so far do not enable us to prove existence of minimizers of $\mathsf{Q} - \mathsf{L}$ using the direct method of the calculus of variations.

It is well known (*cf.* [16] and [17, §68bis]) that condition (15) implies

$$\mathbb{A}(\cdot)\mathbf{v} \otimes \mathbf{w} \cdot \mathbf{v} \otimes \mathbf{w} \geq 0, \quad \text{a.e. in } \Omega, \text{ for every } \mathbf{v}, \mathbf{w} \in \mathbb{R}^3.$$

The converse is not true, as an example in [10] shows. To have existence of minimizers, we are led to replace (15) with a stronger requirement:

$$\int_\Omega \mathbb{A}[\nabla \boldsymbol{\varphi}] \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}^3 \geq c_A \|\boldsymbol{\varphi}\|^2_{H^1(\Omega)} \quad \forall \boldsymbol{\varphi} \in H^1_{0_D}(\Omega; \mathbb{R}^3), \tag{43}$$

for some $c_A > 0$. To obtain (43), it would seem natural to require that $\mathbb{A}(\cdot)$ be uniformly positive, *i.e.*, $\mathbb{A}(\cdot)[\mathbf{A}] \cdot \mathbf{A} > c_A |\mathbf{A}|^2$ a.e. in Ω for all $\mathbf{A} \in \mathbb{R}^{3\times 3}$, for some $c_A > 0$. However, it has been observed in [12, Remark 5.1] that the positivity of $\mathbb{A}$ is valid "only for special classes of $\overset{\circ}{\mathbf{T}}$, which do not include any neighborhood of $\overset{\circ}{\mathbf{T}} = \mathbf{0}$". We now show that if $\mathbb{L}(\cdot)$ is uniformly positive, then the incremental problem admits a unique solution, provided that the "compressions" due to the initial stress are sufficiently small. More precisely, for

$$\overset{\circ}{\tau} := \operatorname*{essinf}_{x \in \Omega} \overset{\circ}{\sigma}(x), \tag{44}$$

with $\overset{\circ}{\sigma}(x)$ the smallest eigenvalue of $\overset{\circ}{\mathbf{T}}(x)$, and for

$$c_K^2 := \sup_{\substack{\boldsymbol{\varphi} \in H^1_{0_D}(\Omega;\mathbb{R}^3) \\ \|\nabla \boldsymbol{\varphi}\|_{L^2(\Omega)} = 1}} \|\mathbf{e}(\boldsymbol{\varphi})\|^2_{L^2(\Omega)}, \tag{45}$$

we have the following

Proposition 1 *Assume* (H1–H3). *Suppose that*

$$\mathbb{L}(\cdot)[\mathbf{A}] \cdot \mathbf{A} \geq c_L |\mathbf{A}|^2 \quad \textit{a.e. in } \Omega, \ \textit{for every } \mathbf{A} \in \mathbb{R}^{3\times 3}_{\mathrm{Sym}}, \tag{46}$$

for some $c_L > 0$, *and that*

$$\overset{\circ}{\tau} + c_L c_K^2 > 0. \tag{47}$$

Then (43) *holds with* $c_A = \overset{\circ}{\tau} + c_L c_K^2$ *and the incremental problem* (42) *has unique solution.*

Proof Let $\boldsymbol{\varphi} \in H^1_{0_D}(\Omega; \mathbb{R}^3)$. Because $\overset{\circ}{\mathbf{T}}(x) \cdot \mathbf{H}^{\mathsf{T}}\mathbf{H} \geq \overset{\circ}{\sigma}(x)|\mathbf{H}|^2$ for all $\mathbf{H} \in \mathbb{R}^{3\times 3}$ (*cf.* [14, Lemma 2]), we have

$$\int_\Omega \overset{\circ}{\mathbf{T}} \cdot \nabla\boldsymbol{\varphi}^{\mathsf{T}} \nabla\boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}^3 \geq \overset{\circ}{\tau} \int_\Omega |\nabla\boldsymbol{\varphi}|^2 \, \mathrm{d}\mathcal{L}^3. \tag{48}$$

By (46) and (45), we have also

$$\int_\Omega \mathbb{L}[\mathbf{e}(\boldsymbol{\varphi})] \cdot \mathbf{e}(\boldsymbol{\varphi}) \, \mathrm{d}\mathcal{L}^3 \geq c_L c_K^2 \int_\Omega |\nabla\boldsymbol{\varphi}|^2 \, \mathrm{d}\mathcal{L}^3. \tag{49}$$

From (11), (48), and (49), we obtain (43) with $c_A = \overset{\circ}{\tau} + c_L c_K^2$. Then, the functional $\mathsf{Q} - \mathsf{L}$ is coercive in $H^1_{0_D}(\Omega; \mathbb{R}^3)$. By Lemma 3, the functional $\mathsf{Q} - \mathsf{L}$ is also lower-semicontinuous with respect to the weak-H^1 convergence. Using the direct method of the calculus of variations, we establish existence of a solution to the incremental problem (42). This solution is unique because $\mathsf{Q} - \mathsf{L}$ is quadratic and, by (43), positive definite. □

Remark 2 In view of (11), condition (15) reads

$$\int_\Omega \overset{\circ}{\mathbf{T}} \cdot \nabla\boldsymbol{\varphi}^{\mathsf{T}} \nabla\boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}^3 + \int_\Omega \mathbb{L}[\mathbf{e}(\boldsymbol{\varphi})] \cdot \mathbf{e}(\boldsymbol{\varphi}) \, \mathrm{d}\mathcal{L}^3 \geq 0. \tag{50}$$

If (46) holds, then the second term on the left-hand side of (50), which represents the effect of elastic response, is always positive, and hence has a stabilizing effect. If the initial stress is *compressive*, that is to say, the smallest eigenvalue of $\overset{\circ}{\mathbf{T}}$ attains negative values, the first term on the left-hand side of (50) can be negative, and hence it may have a destabilizing effect. Conditions (46) and (47) have the following mechanical interpretation (*cf.* [6]): that the destabilizing effect of the initial stress must be compensated by the stabilizing effect on the elastic response.

To obtain compactness of almost minimizers, it is natural to assume that the energy density has a quadratic growth "from the reference configuration". Indeed, it suffices to assume a quadratic growth for small strains and a linear growth for large. More precisely, we assume that

(H5) there exists $\rho > 0$ and $c_L > 0$ such that, for

$$\psi(t) = \begin{cases} c_L t^2/2 & \text{if } t \leq \rho, \\ c_L(\rho t - \rho^2/2) & \text{if } t > \rho, \end{cases} \tag{51}$$

the stored energy w satisfies

$$\mathsf{w}(x, \mathbf{E}) - \mathsf{w}(x, \mathbf{0}) - \partial\mathsf{w}(x, \mathbf{0}) \cdot \mathbf{E} \geq \psi(|\mathbf{E}|)$$

for almost all x in Ω and for all $\mathbf{E}$ in $\mathbb{R}^{3\times 3}_{\mathrm{Sym}}$.

For the proof of the next proposition, see [2, Prop. 3.4].

Proposition 2 *Let ψ be as in (51). Then there exists a constant $\gamma > 0$ such that*

$$\gamma \int_{\Omega} |\nabla \boldsymbol{\varphi}|^2 \, d\mathcal{L}^3 \leq \int_{\Omega} \psi(|\mathbf{E}(\nabla \boldsymbol{\varphi})|) \, d\mathcal{L}^3, \tag{52}$$

for every $\boldsymbol{\varphi} \in H^1_{0_D}(\Omega; \mathbb{R}^3)$ satisfying $\mathrm{essinf}_{\Omega} \det(\mathbf{I} + \nabla \boldsymbol{\varphi}) > 0$.

We introduce the following hypothesis on the quantity $\mathring{\tau}$ defined in (44).

(H6) Let γ be as in Proposition 2. Then,

$$\mathring{\tau} > -2\gamma.$$

Proposition 3 *Assume* (H1–H3) *and* (H5–H6). *Then, (46) and (47) hold.*

Proof As already observed in Sect. 2, hypotheses (H1–H3) imply that the function $\mathbf{A} \mapsto \mathsf{w}(x, \mathbf{A})$ is differentiable near the origin for a.e. $x \in \Omega$. Thus, for a.e. $x \in \Omega$ and for every $\eta > 0$ sufficiently small there exists $\rho_\eta(x) \in (0, 1)$ such that

$$\frac{1}{2} \partial^2 \mathsf{w}(x, \rho_\eta(x) \eta \mathbf{A})[\eta \mathbf{A}] \cdot \eta \mathbf{A} = \mathsf{w}(x, \eta \mathbf{A}) - \mathsf{w}(x, 0) - \partial \mathsf{w}(x, 0) \cdot \eta \mathbf{A},$$

and by (H5) we have

$$\partial^2 \mathsf{w}(x, \rho_\eta(x) \eta \mathbf{A})[\mathbf{A}] \cdot \mathbf{A} \geq c_L |\mathbf{A}|^2.$$

On letting $\eta \to 0$ we obtain

$$\mathbb{L}(x)[\mathbf{A}] \cdot \mathbf{A} := \partial^2 \mathsf{w}(x, 0)[\mathbf{A}] \cdot \mathbf{A} \geq c_L |\mathbf{A}|^2,$$

which is (46). Next, take $\boldsymbol{\varphi} \in C^\infty(\Omega; \mathbb{R}^3)$ such that $\boldsymbol{\varphi} = \mathbf{0}$ on $\partial_D \Omega$, $\det(\mathbf{I} + \nabla \boldsymbol{\varphi}) > 0$ in Ω, and $\int_\Omega |\nabla \boldsymbol{\varphi}| \, \mathrm{d}\mathcal{L}^3 = 1$. It follows from (52) that: $\gamma \leq \varepsilon^{-2} \int_\Omega \psi(|\mathbf{E}(\varepsilon \nabla \boldsymbol{\varphi})|) \, \mathrm{d}\mathcal{L}^3$. Letting $\varepsilon \to 0$, by bounded convergence we obtain $\gamma \leq \frac{c_L}{2} \int_\Omega |\mathbf{e}(\boldsymbol{\varphi})|^2 \, \mathrm{d}\mathcal{L}^3$, and from (45) we conclude:

$$\gamma \leq \frac{c_L c_K^2}{2},$$

This inequality, combined with (H6), gives (47). □

We are now ready to state a result concerning convergence of almost minimizers.

Theorem 2 *Assume that* (H1–H3) *and* (H5–H6) *hold. Let $\{\mathbf{u}_\varepsilon\} \subset H^1_{0_D}(\Omega; \mathbb{R}^3)$ be a collection of almost minimizers, in the sense of (41). Then, as ε tends to zero, $\mathbf{u}_\varepsilon$ converges weakly to $\mathbf{u}$ in $H^1_{0_D}(\Omega; \mathbb{R}^3)$, and $\mathbf{u}$ is the unique minimizer of $\mathsf{Q}(\cdot) - \mathsf{L}[\cdot]$.*

Proof We have:

$$\begin{aligned}
\frac{\mathsf{E}(0, \varepsilon \mathbf{u}_\varepsilon) - \mathsf{E}(0, \mathbf{0})}{\varepsilon^2} &= \frac{1}{\varepsilon^2} \int_\Omega \mathsf{w}(\cdot, \mathbf{E}(\varepsilon \mathbf{u}_\varepsilon)) - \mathsf{w}(\cdot, \mathbf{0}) \, \mathrm{d}\mathcal{L}^3 - \frac{\mathring{\mathsf{L}}[\mathbf{u}_\varepsilon]}{\varepsilon} \\
&= \frac{1}{\varepsilon^2} \int_\Omega \mathsf{w}(\cdot, \mathbf{E}(\varepsilon \mathbf{u}_\varepsilon)) - \mathsf{w}(\cdot, \mathbf{0}) - \mathring{\mathbf{T}} \cdot (\varepsilon \nabla \mathbf{u}_\varepsilon) \, \mathrm{d}\mathcal{L}^3 \\
&= \frac{1}{\varepsilon^2} \int_\Omega \mathsf{w}(\cdot, \mathbf{E}(\varepsilon \mathbf{u}_\varepsilon)) - \mathsf{w}(\cdot, \mathbf{0}) - \mathring{\mathbf{T}} \cdot \left(\mathbf{E}(\varepsilon \nabla \mathbf{u}_\varepsilon) - \frac{\varepsilon^2}{2} \nabla \mathbf{u}_\varepsilon^{\mathsf{T}} \nabla \mathbf{u}_\varepsilon \right) \mathrm{d}\mathcal{L}^3 \\
&\geq \frac{1}{\varepsilon^2} \int_\Omega \psi(|\mathbf{E}(\varepsilon \mathbf{u}_\varepsilon)|) \, \mathrm{d}\mathcal{L}^3 + \frac{1}{2} \mathring{\tau} \int_\Omega |\nabla \mathbf{u}_\varepsilon|^2 \, \mathrm{d}\mathcal{L}^3 \\
&\geq \left(\gamma + \frac{\mathring{\tau}}{2} \right) \int_\Omega |\nabla \mathbf{u}_\varepsilon|^2 \, \mathrm{d}\mathcal{L}^3,
\end{aligned} \tag{53}$$

where the last two inequalities follow from (H5), (48) and (52).

By (41) and (53), we have for ε sufficiently small,

$$\begin{aligned} 1 \geq \frac{\mathsf{E}(\varepsilon, \varepsilon \mathbf{u}_\varepsilon) - \inf_{\mathbf{u} \in H^1_{0_D}(\Omega;\mathbb{R}^3)} \mathsf{E}(\varepsilon, \varepsilon \mathbf{u})}{\varepsilon^2} &\geq \frac{\mathsf{E}(\varepsilon, \varepsilon \mathbf{u}_\varepsilon) - \mathsf{E}(\varepsilon, \mathbf{0})}{\varepsilon^2} \\ &= \frac{\mathsf{E}(0, \varepsilon \mathbf{u}_\varepsilon) - \mathsf{E}(0, \mathbf{0})}{\varepsilon^2} - \mathsf{L}[\mathbf{u}_\varepsilon] \\ &\geq \left(\gamma + \frac{\mathring{\tau}}{2}\right) \int_\Omega |\nabla \mathbf{u}_\varepsilon|^2 \, d\mathcal{L}^3 - \mathsf{L}[\mathbf{u}_\varepsilon]. \end{aligned} \tag{54}$$

Since $\mathbf{d} \in L^2(\Omega; \mathbb{R}^3)$ and $\mathbf{c} \in L^2(\partial_N \Omega; \mathbb{R}^3)$, for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$\mathsf{L}[\mathbf{u}_\varepsilon] \leq C_\delta + \delta \int_\Omega |\nabla \mathbf{u}_\varepsilon|^2 \, d\mathcal{L}^3, \tag{55}$$

and, by (54) and (55),

$$1 + C_\delta \geq \left(\gamma + \frac{\mathring{\tau}}{2} - \delta\right) \int_\Omega |\nabla \mathbf{u}_\varepsilon|^2 \, d\mathcal{L}^3;$$

thus, by (H6) and by Poincaré's inequality, $\{\mathbf{u}_\varepsilon\}$ is bounded in $H^1(\Omega; \mathbb{R}^3)$. Hence, there is a sequence $\{\varepsilon_k\} \subset \mathbb{R}^+$ approaching zero and $\mathbf{u} \in H^1_{0_D}(\Omega; \mathbb{R}^3)$ such that $\mathbf{u}_{\varepsilon_k}$ converges weakly to $\mathbf{u}$ in $H^1_{0_D}(\Omega; \mathbb{R}^3)$ as $k \to +\infty$. Since $\partial_D \Omega$ has positive measure, it follows from (53) that $\mathbf{u} = \mathbf{0}$ is a H^1-local minimizer of $\mathsf{E}(0; \cdot)$ over $H^1_{0_D}(\Omega; \mathbb{R}^3)$, thus (H4) holds true by Lemma 1 and hence Theorem 1 applies. Given any $\bar{\mathbf{u}} \in H^1_{0_D}(\Omega; \mathbb{R}^3)$, it follows from Theorem 1 that (see also [1,3]):

$$\begin{aligned} \mathsf{Q}(\mathbf{u}) - \mathsf{L}(\mathbf{u}) &\leq \liminf_{k \to \infty} \frac{\mathsf{E}(\varepsilon_k; \varepsilon_k \mathbf{u}_{\varepsilon_k}) - \mathsf{E}(0; \mathbf{0})}{\varepsilon_k^2} \\ &\leq \limsup_{k \to \infty} \frac{\mathsf{E}(\varepsilon_k; \varepsilon_k \bar{\mathbf{u}}_{\varepsilon_k}) - \mathsf{E}(0; \mathbf{0})}{\varepsilon_k^2} \leq \mathsf{Q}(\bar{\mathbf{u}}) - \mathsf{L}(\bar{\mathbf{u}}), \end{aligned}$$

where $\{\bar{\mathbf{u}}_{\varepsilon_k}\}$ is a recovery sequence for $\bar{\mathbf{u}}$; thus $\mathbf{u}$ solves the incremental problem. By Proposition 1 and Proposition 3, $\mathbf{u}$ is the unique solution. Since $\mathbf{u}$ does not depend on the particular sequence, we conclude that the whole family $\{\mathbf{u}_\varepsilon\}$ converges weakly in $H^1_{0_D}(\Omega; \mathbb{R}^3)$ to $\mathbf{u}$ as $\varepsilon \to 0$. □

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