As for the tensor trajectories [14] the concept of the peripherality and the validity of eq. (6) are still an open question. We have also to answer the question as to whether or not the HISH can be generalized to the double helicity-flip amplitudes.

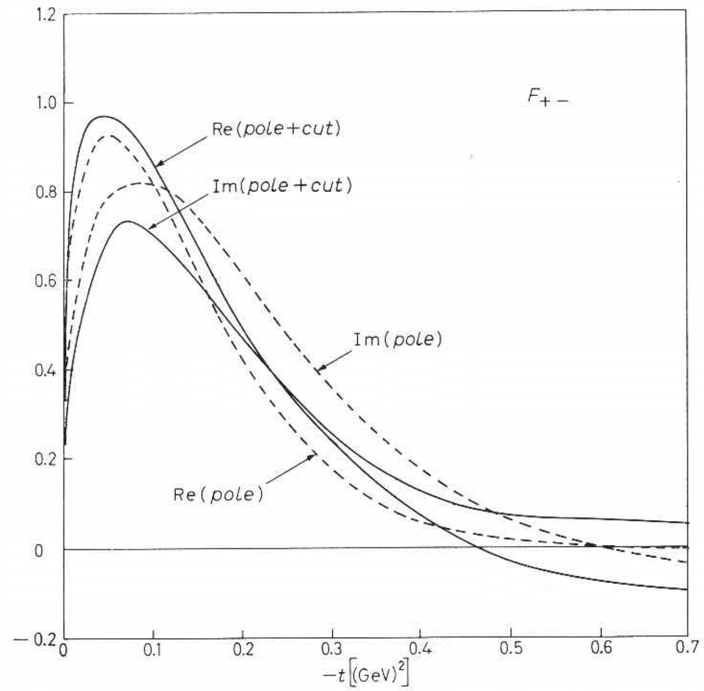

Fig. 2. – Theoretical curves for helicity-flip amplitudes. The parameters are the same as in Fig. 1.

It should be remarked that the physical meaning of the constants $c(b)$ and s are not self-evident. The constant c (independent of b) will have an interrelation with the F/D ratio which is known to stay approximately constant in the s-channel helicity amplitudes [8,14].

We conclude that the helicity-independent structure hypothesis works fairly well in the ρ exchange process at intermediate energies and that the peripherality of the helicity-nonflip amplitude is not the consequence of the absorption. In fact the cross-over zero has been reproduced by the HISH together with the NWSZ in the residue of the helicity-flip amplitude. It would be worthy to emphasize that the absorption was necessary only to remedy difficulties in the phase of the Regge-pole amplitudes.

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One of us (H.I.) would like to express his thanks to Iwanami Fûjukai for the financial support.

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(14) V. Barger, K. Geer and F. Halzen: *Nucl. Phys.*, **49** B, 302 (1972).

LETTERE AL NUOVO CIMENTO VOL. 8, N. 9 27 Ottobre 1973

On the Connection between the Probabilistic and the Hilbert-Space Description of Dynamical Systems.

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(ricevuto il 9 Marzo 1973)

Let ξ be a system whose (measurable) phase space we denote by $(\Omega,\mathscr{B})$. The classical (Schrödinger) description of the system is completely accomplished by its trajectory in the following sense: there is a 1-1 correspondence between the set of all measurement operations on the system and an Abelian algebra $\mathscr{A}$ of real measurable functions on $(\Omega,\mathscr{B})$ (*). The content of the theory is specified by this correspondence; its aim is the computation of the values $f(\xi_t)$, for f in $\mathscr{A}$ and $t\in\mathrm{R}$, which are the results of the measurement operation corresponding to f, performed on the system at the moment t. The transition to the Heisenberg point of view needs a further hypothesis, namely: the state of the system at the time t is uniquely determined by its state at an arbitrary time t_0, by

$$\xi_t = T^t_{t_0}\xi_{t_0}\,, \qquad T^t_{t_0}\in \mathrm{Aut}\,(\Omega,\mathscr{B})\,, \tag{1.1}$$

where Aut $(\Omega,\mathscr{B})$ is the group of automorphisms (1-1, bimeasurable mappings) of $(\Omega,\mathscr{B})$.

A system ξ which enjoys this property is called a (measurable) dynamical system. With the notation $f(\xi)=\hat{\xi}(f)$, $\hat{\hat{T}}(f)=f\circ\overset{-1}{T}$, $T\in\mathrm{Aut}\,(\Omega,\mathscr{B})$ the theory purports to compute the values

$$\hat{\xi}_t(f)=\hat{\xi}_{t_0}(\hat{\hat{T}}{}^{t_0}_t f)\,, \qquad f\in\mathscr{A}\,. \tag{1.2}$$

Here the $\hat{\xi}$'s are characters of $\mathscr{A}$, so that knowledge of the expressions (1.2) is equivalent to knowledge of the expressions

$$\hat{\xi}_{t_0}[(\hat{\hat{T}}{}^{t_0}_{t_1}f_1)\cdot\ldots\cdot(\hat{\hat{T}}{}^{t_0}_{t_n}f_n)]\,, \qquad f_\iota\in\mathscr{A},\ 1\leqslant\iota\leqslant n\,. \tag{1.3}$$

(*) To a physicist, the introduction of the space of states as a noumenic entity separated by the (phenomenic) algebra of the observables may seem awkward, but reasons of clarity—mathematically supported by Gel'fand's isomorphism theorem—suggest this approach.

We call these expressions the « Green's functions » of the system ξ. In a probabilistic approach to the description of a system one is no longer concerned with values $f(\xi)$, but with expectation values

$$\hat{\xi}(f)=\int_\Omega f\,d\xi=\int_\Omega d[M(f)\xi]\,, \tag{1.4}$$

where, now, ξ is a probability measure on $(\Omega,\mathscr{B})$ and $M(f)$ denotes the multiplication operator by f. Equations (1.1), (1.2), (1.3) can be kept in this new interpretation with the further condition that operator $T^{t}_{t_0}$, in (1.1), maps probability measures into probability measures.

This last condition, plus linearity, defines a Markov operator (cf. ([1])). In such case the system is called a « Markov system » and the (semi)-group equation for $T^{t}_{t_0}$ which follows from uniqueness is the well-known Chapman-Kolmogoroff equation. The simplest Markov operators are those defined by the equation

$$\hat{T}(\xi)=\xi\circ \overset{-1}{T}\,, \qquad T\in \mathrm{Aut}\,(\Omega,\mathscr{B})\,. \tag{1.5}$$

In this note we shall consider only such operators (*), *i.e.* the probabilistic description of systems with a deterministic evolution. Also in such circumstances the theory aims at computing expressions (1.4), which are now no longer equivalent to (1.3) because of the loss of multiplicativity. If we write (1.3) in terms of (1.4) and perform the « change of reference » $Q\in \mathrm{Aut}\,(\Omega,\mathscr{B})$ (**) at time t_n, (1.3) becomes

$$\{M(f_{t_n})\cdot\hat{Q}[M(f_{t_{n-1}})\cdot\ldots\cdot M(f_{t_1})\,\hat{\xi}_0]\}(\Omega) \tag{1.6}$$

$(t_0=0,\ f_{t_i}=T^{0}_{t_i}f_i)$; this is no longer commutative in the $M(f_{t_i})$'s. So we fix the order $t_1\leqslant t_2\leqslant\ldots\leqslant t_n$ in the expression for the Green functions and interpret them as joint expectation values of the observables f_i at time t_i taken in the reference frame R_i.

Both expressions (1.3) and (1.6) resemble now the well-known ones

$$\langle\Omega_0;\,\varphi_{t_1}(f_1')\cdot\ldots\cdot\varphi_{t_n}(f_n')\,\Omega_0\rangle\,, \tag{1.7}$$

which give the vacuum expectation value of a time-ordered product of field operators. The main difference is that (1.3) is a *real* expression, *linear* in $\hat{\xi}_0$, while (1.7) is a *complex* expression *quadratic* in Ω_0. Thus one could be tempted to write, in a purely formal way, expression (1.3) (respectively (1.6)) in the form

$$\langle\sqrt{\xi_0};\,\theta_{t_1}(f_1)\cdot\ldots\cdot\theta_{t_n}(f_n)\cdot\sqrt{\xi_0}\rangle \tag{1.8}$$

and to interpret $\sqrt{\xi_0}$ as a vector in a certain Hilbert space, and the $\theta_{t_i}(f_i)$'s as linear operators therein.

([1]) E. Hopf: *Journ. Rat. Mech. Anal.*, **3**, 13 (1954).

(*) That the general case can, in most circumstances, be reduced to this follows from the results of Verchik ([2]).

([2]) A. M. Verchik: *Isv. Acad. Nauk SSSR, Ser. Mat.*, **29**, 127 (1965).

(**) Following the indications of ([3]) we try to keep as general as possible the group of transformations of reference.

([3]) F. J. Dyson: *Bull. Amer. Math. Soc.*, **78**, 635 (1972).

The results listed below show that there is a natural way to give a rigorous meaning to expression like (1.8), *i.e.* that it is possible to define an operational calculus in which « square roots of measures » correspond to elements in a Hilbert space. The main features of this correspondence are the following:

Banach space $\mathscr{M}$ of bounded measures	$\Rightarrow$	Hilbert space $\mathscr{H}$ of « square roots of measures »,
Markov action of $\mathrm{Aut}\,(\Omega,\mathscr{B})$ on $\mathscr{M}$	$\Rightarrow$	real-unitary action of $\mathrm{Aut}\,(\Omega,\mathscr{B})$ on $\mathscr{H}$,
Abelian algebra of multiplication operators by functions in $L^\infty(\Omega,\mathscr{B})$, on $\mathscr{M}$	$\Rightarrow$	« locally » maximal Abelian algebra $\mathscr{A}$ of operators on $\mathscr{H}$.

Furthermore one has (*) « locally » (cf. Proposition 6 below):

$$\mathscr{A}\otimes \mathrm{Aut}\,(\Omega,\mathscr{B})=\mathscr{B}(\mathscr{H})\,.$$

These features suggest a deep connection between the present work and recent ideas in quantum field theory (cf. ([4-6]) for example). This connection will be discussed elsewhere at length. We now list some results.

Let $(\Omega,\mathscr{B})$ denote a measurable space and $\mathscr{M}(\Omega,\mathscr{B})$ the Banach space of bounded real measures on $(\Omega,\mathscr{B})$ endowed with the « total variation norm ». We write $x\perp y$ for two disjoint measures, and $\mathscr{M}^+(\Omega,\mathscr{B})$ for the cone of positive measures.

Theorem 1. There exists a Hilbert space $\mathscr{H}(\Omega,\mathscr{B})$ and an homeomorphism $\alpha:\mathscr{M}(\Omega,\mathscr{B})\to\mathscr{H}(\Omega,\mathscr{B})$ enjoying the following properties:

1) $\|x\|_{\mathscr{M}(\Omega,\mathscr{B})}=\|\alpha(x)\|^2_{\mathscr{H}(\Omega,\mathscr{B})}$, $\qquad\forall x\in\mathscr{M}(\Omega,\mathscr{B})$,

2) $\langle\alpha(x);\alpha(y)\rangle=0$, if and only if $x\perp y$, $\forall x,y\in\mathscr{M}^+(\Omega,\mathscr{B})$,

3) $\alpha(x+y)=\alpha(x)+\alpha(y)$, if and only if $x\perp y$, $\forall x,y\in\mathscr{M}(\Omega,\mathscr{B})$,

4) $\alpha(\lambda x)=\sqrt{\lambda}\cdot\alpha(x)$, $\lambda\in\mathsf{R}^+$, $x\in\mathscr{M}(\Omega,\mathscr{B})$,

5) $\alpha(-x)=-\alpha(x)$, $x\in\mathscr{M}(\Omega,\mathscr{B})$,

6) $\alpha(m)\pm\alpha(n)=\alpha[(\chi_+\pm\chi_-)\cdot(m+n\pm2\sqrt{mn})$, $m,n\geqslant0$; χ_+, χ_- are the characteristic functions of a Jordan partition of Ω relative to $m-n$.

The properties listed in Theorem 1, characterize $\mathscr{H}(\Omega,\mathscr{B})$ in the sense specified by the following theorem.

Theorem 2. If H is an Hilbert space and $\beta:\mathscr{M}(\Omega,\mathscr{B})\to H$ is an homeomorphism which satisfies conditions 1)-5) of Theorem 1, then there exists a unitary isomorphism $u:\mathscr{H}(\Omega,\mathscr{B})\to H$.

The algebra $L^\infty(\Omega,\mathscr{B})$ of bounded measurable real functions on $(\Omega,\mathscr{B})$ acts naturally on $\mathscr{M}(\Omega,\mathscr{B})$ by multiplication: $M(f)(x)=f\cdot x$.

(*) We tend to interpret $\mathscr{A}$ as the algebra of (classical) observables and $\mathrm{Aut}\,(\Omega,\mathscr{B})$ as the group of change of reference.

([4]) I. E. Segal: *Trans. Amer. Math. Soc.*, **81**, 106 (1956).

([5]) E. Nelson: preprint.

([6]) F. Guerra, L. Rosen and B. Simon: preprint.

Theorem 3. The natural action of $L^\infty(\Omega,\mathscr{B})$ on $\mathscr{M}(\Omega,\mathscr{B})$ induces an action $\tilde{M}$ of this algebra on $\mathscr{H}(\Omega,\mathscr{B})$ which satisfies the following properties:

1) $\tilde{M}(f\cdot g) \quad = \tilde{M}(f)\cdot\tilde{M}(g)\,, \qquad f,g\in L^\infty(\Omega,\mathscr{B})\,,$

2) $\tilde{M}(\lambda\cdot f) \quad = \sqrt{\lambda}\cdot\tilde{M}(f)\,, \qquad \lambda\in \mathsf{R}^+\,, \qquad f\in L^\infty(\Omega,\mathscr{B})\,,$

3) $\tilde{M}(f+g) \quad = \tilde{M}(f)+\tilde{M}(g)\,,$ if and only if $f\perp g$,

4) $\sum_{i=1}^{k}\varepsilon_i\cdot\tilde{M}(f_i)=\tilde{M}\left[\left\{\left(\sum_{i=1}^{k}\varepsilon_i\cdot\sqrt{f_i}\right)^+\right\}^2-\left\{\left(\sum_{i=1}^{k}\varepsilon_i\cdot\sqrt{f_i}\right)^-\right\}^2\right]$ for every $k\in\mathsf{N}$, $\varepsilon_i=\pm1$ and $f_i\in L^\infty_+(\Omega,\mathscr{B})$, $1\leqslant i\leqslant k$.

Each $\tilde{M}(f)$ is a linear, bounded operator on $\mathscr{H}(\Omega,\mathscr{B})$ and $\tilde{M}(L^\infty(\Omega,\mathscr{B}))=\mathscr{A}$ is a Abelian algebra.

Now for $x\in\mathscr{M}(\Omega,\mathscr{B})$, put

$$\mathscr{H}_x(\Omega,\mathscr{B})=\overline{\mathscr{A}[\alpha(x)]}\,,$$

i.e. $\mathscr{H}_x(\Omega,\mathscr{B})$ is the norm-closure of the orbit of $\alpha(x)$ by the action of $\mathscr{A}$.

Theorem 4.

$$\mathscr{H}_x(\Omega,\mathscr{B})=\alpha(\mathscr{M}_x)\,,$$

where $\mathscr{M}_x$ denotes the subspace of all measures $y<x$. Furthermore, the following isomorphisms take place:

$$\mathscr{H}_x(\Omega,\mathscr{B})\approx L^2(\Omega,\mathscr{B},|x|)\approx L^2(\mathscr{A};\tau_x)\,,$$

where $L^2(\mathscr{A};\tau_x)$ denotes the space obtained applying the Gel'fand-Naimark-Segal construction to the algebra $\mathscr{A}$ and the state $a\mapsto\langle\alpha(x);a\cdot\alpha(x)\rangle$ (we suppose $\|x\|=1$). In particular $\mathscr{H}_x$ coincides with $\mathscr{H}_y$ if and only if $x\sim y$; and $\mathscr{H}_x$ is orthogonal to $\mathscr{H}_y$ if and only if $x\perp y$ (*).

The group $\mathrm{Aut}(\Omega,\mathscr{B})$ acts naturally on $\mathscr{M}(\Omega,\mathscr{B})$ by means of the formula

$$\hat{T}(x)=x\circ\overset{-1}{T}\,,\qquad\qquad T\in\mathrm{Aut}(\Omega,\mathscr{B})\,.$$

The operators $\hat{T}$ are Markov operators in the sense of (1).

Theorem 5. The Markov action $\hat{T}$ of $\mathrm{Aut}(\Omega,\mathscr{B})$ on $\mathscr{M}(\Omega,\mathscr{B})$ induces a unitary action $\tilde{T}$ of this group on $\mathscr{H}(\Omega,\mathscr{B})$.

Proposition 6. The cross-product $\mathscr{A}_x\otimes G_x$ is the whole space $\mathscr{B}(\mathscr{H}_x)$.

Proposition 7. For every $x\in\mathscr{M}(\Omega,\mathscr{B})$ one has

$$\tilde{T}(\mathscr{H}_x)=\mathscr{H}_{\hat{T}x}\,,$$

(*) Thus the space $\mathscr{H}_x(\Omega,B)$ depends only on the equivalence class of x and is isomorphic to the «intrinsic Hilbert space» of MACKEY (7).

(7) G. W. MACKEY: *Mathematical Foundations of Quantum Mechanics* (New York, 1963).

in particular the subgroup G_x which leaves $\mathscr{H}_x$ invariant is the image of the subgroup of $\mathrm{Aut}(\Omega,\mathscr{B})$ which leaves x quasi-invariant.

Let now $(\Omega,\mathscr{B})\approx(S\times G,\mathcal{B}\times\mathcal{D})$, where G is a countable group with a measurable action U on $(S,\mathcal{B})$. Let $x\in\mathscr{M}^+(\Omega,\mathscr{B})$ be an U-almost invariant measure and $\nu\in\mathscr{M}(G,\mathcal{D})$ a bounded measure equivalent to the Haar measure on G. Denote by R the right translation on G. Then,

Proposition 8. Consider the unitary representation $\tilde{T}=\tilde{U}\otimes\tilde{R}$ defined on $\mathscr{H}(S\times G;\mathcal{B}\times\mathcal{D})$ as in Theorem 5 and the group $K=(\tilde{U}\otimes\tilde{R})(G)$. Let $\mathscr{A}_S$ denote the Abelian von Neumann algebra which is the image of $L^\infty(S,\mathcal{B})$—considered as a sub-algebra of $L^\infty(S\times G;\mathcal{B}\times\mathcal{D})$ in the isomorphism defined in Theorem 3.

Then, the crossed product $\mathscr{A}_S\otimes K$ is isomorphic with the ring $\mathscr{D}$ obtained from $L^2(S,\mathcal{B},x)$ and G applying the group-measure space construction (cf. (8)).

Conclusions. – The approach outlined here to the study of dynamical systems (*i.e.* subgroups of $\mathrm{Aut}(\Omega,\mathscr{B})$) is essentially a generalization of the «Koopman program» (cf. (9)), to which it reduces when one considers the restrictions of these groups on the «fibers» $\mathscr{H}_x$. It is also a natural context for the introduction of the «Markov property» and for the proof of an assertion of VON NEUMANN. The proofs will be reported elsewhere (10).

(8) J. DIXMIER: *Les algèbres d'opérateurs dans l'espace hilbertien* (Paris, 1969).

(9) J. VON NEUMANN: *Collected Works*, Vol. **2**, p. 307.

(10) L. ACCARDI: *On square roots of measures*, in *Proc. of the 1973 Varenna Summer School*, to appear.